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EFFECT OF SPATIAL DECORRELATION ON NULLING PERFORMANCE OF LINEAR ADAPTIVE ARRAY

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The presence of a disturbed ionosphere can cause signal decorrelations in time, frequency, and space. This report is concerned with the impact of spatial decorrelations on the performance of an adaptive array used to null interfering sources and jammers. Attention is concentrated on the effect of spatial decorrelation between spaced broadbeam antenna elements rather than decorrelation across individual antenna apertures. While fully adaptive arrays are considered, more analysis and evaluation is carried out for the

20. ABSTRACT (Continued)

sidelobe canceller (SLC) configuration. Complex Gaussian channel fluctuations are assumed for the disturbed ionosphere.

Three types of output SNR analyses are conducted: non-fading, fading with array time constants short enough to adapt to fading, and fading with array time constants too slow to adapt to fading. Most detailed analytical results are carried out for one or two antenna elements and slow fading time constants. However, some more general results on processing gain are obtained for a linear array of M SLC antennas and slow fading time constants where processing gain is defined here as the ratio of output SNR with the array functioning to achieve optimum performance to output SNR with no adaptive array. Curves and simple analytic expressions are given relating processing gain to a suitable decorrelation parameter. The latter, in turn, is related to the rms angle of arrival fluctuations.

Error rate expressions are developed for coherent and incoherent FSK, PSK, and DPSK modems when used in conjunction with an SLC adaptive array. While general analytical results are obtained, numerical results are obtained for the simplest case of a single SLC antenna and a single jammer.

TABLE OF CONTENTS

TABLE OF CONTENTS
(Concluded)

LIST OF ILLUSTRATIONS

SECTION I

EFFECT OF A SPACE/TIME VARIANT CHANNEL ON ADAPTIVE ARRAY PERFORMANCE: SPATIAL DECORRELATION EFFECTS

This document is a topical report under Contract DNA001-82-C-0273 covering the period June 21, 1982 to March 15, 1983. The subject matter concerns the effect of a space/time variant channel on the ability of an adaptive array to null jammers. Attention is concentrated on the effect of spatial decorrelation between antenna elements (the elements themselves are assumed small enough not to be affected by spatial decorrelation).

While other adaptive array types are considered, more analysis and evaluation is carried out for the sidelobe canceller (SLC) configuration. For detailed performance evaluation a Gaussian spatial correlation function and complex Gaussian statistics are assumed for channel fluctuations. These assumptions are appropriate to transionospheric propagation with saturated scintillation.

1.1 CONTENTS

The subsequent report is divided into seven sections and one appendix. Section 2 presents the configurations for three basic types of adaptive arrays: the fully adaptive array, the multiple beam array, and the sidelobe canceller.

Section 3 presents the equations defining the weights for optimum steady state performance and the corresponding mean squared error, signal-to-noise ratio, and antenna pattern. The space/time channel averages and statistics needed to calculate performance are presented.

Section 4 presents detailed calculations of optimum performance for the case of a two-element adaptive array. As in the other calculations in this report, only narrowband signals are assumed so that no degradations due to frequency-selective distortion are calculated. This was done to place emphasis on the impact of spatial selectivity (i.e., decorrelation) or direction-of-arrival fluctuations upon performance. The space/time variant channel model without frequency selectivity consists of a collection of channels from each transmitter to each antenna element with each such channel modeled as a complex time-variant multiplier. Three types of analyses are conducted: non-fading, fading with array time constants short enough to adapt to fading, fading with array time constants too slow to adapt to fading. A two-element half wavelength-spaced array is considered and response to a single jammer evaluated.

Section 5 in large part presents detailed calculations for the SLC (sidelobe canceller) array that parallel those in Section 4 for the fully adaptive array. In this case, however, a linear array with a single antenna and two auxiliary antennas is considered together with the response to two jammers. At the end of this section some analytical results on processing gain against a single jammer at large J/N (jammer-to-noise) ratios are obtained for a linear array of M SLC auxiliary antennas. By processing gain we mean the ratio of output SNR with the array functioning to achieve optimum performance to output SNR with no adaptive array.

Section 6 carries out further analysis of SLC processing gain for a single jammer, a linear array of M auxiliary antennas, and array time constants long compared to the fading time constants. With the aid of an appropriate spatial correlation function, formulae are obtained for processing gain. Plots are presented of processing gain as a function of the decorrelation between fading on adjacent antennas.

Section 7 presents the basic analysis for the more general case of a three-dimensional grid of SLC auxiliary antennas and multiple jammers. A computer program was written to calculate processing gain for this general case but it has not been exercised to any extent yet.

Section 8 develops expressions for the error rate of modems used in conjunction with an SLC (sidelobe canceller) adaptive array. It is assumed that the propagation medium produces decorrelations in time, space, and frequency, on the transmitted signals. However, it is assumed that the adaptive array time constants are too slow to adapt to the fading and that the signal bandwidth is narrow enough to avoid significant frequency-selective fading. Thus the performance degradation suffered by the array is caused by spatial selectivity and nonadaptation to the fading. The analysis is concerned with the error rate performance of a digital modem connected to the output after the SLC has attempted to cancel the jamming signals. As in the case of the adaptive array, two cases may be considered: the modem time constants may or may not allow adaptation to the fading. In this section we consider the case of a fading adaptive modem. In order to obtain adequate error rate performance it is necessary to employ coding, interleaving, and forward error correction techniques. The raw error rates computed here may be used to evaluate the improvement offered by interleaved hard-decision error correction coding techniques. Some numerical results on error rate are obtained showing the impact of spatial decorrelation.

1.2 CONCLUSIONS

In the case of a fading adaptive array, it is possible in theory * for the antenna pattern nulls to become sharper with increasing jammers' power until the jammers are completely nulled and the output SNR is independent of jammer levels. This same behavior exists for the non-fading non-dispersive channel. However, there are degradations over the non-fading case even with a fading adaptive array. Aside from fading per se, which produces large modem performance degradations over the non-fading case, there is degradation due to the fact that the fluctuating angle of arrival of the signal and the jammers may cause jammers to apparently come close enough to the signal direction to prevent nulling at any reasonable jammer level.

In the case of the fading non-adaptive array, the situation is quite different. For the fully adaptive array it is shown that as the fading becomes severe enough so that the mean field disappears, the array "shuts down", i.e., the adaptive weights go to zero. With any non-zero mean field the output SNR at large jammer levels becomes proportional to the input J/S ratio, indicating that a perfect null cannot be formed at large jammer powers as in the fading adaptive case. The sidelobe canceller does not shut down as the mean field goes to zero. However, the output SNR is proportional to the J/S ratios at large jammer levels also. If the decorrelation between auxiliary antennas is sufficiently small and if sufficient number of antennas exist in relation to the number of jammers, the SLC array can provide adequate processing gain against the jammers. The reader is referred to Figures 6.2 and 8.2 for calculations of the impact of spatial decorrelation on processing gain

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* Practical limitations prevent reaching complete jammer cancellation. We do not discuss these limitations here since our emphasis is on determining the impact of spatial decorrelation.

and error rate for the case of a single jammer. With only one auxiliary antenna the impact of spatial decorrelation becomes significant with small amounts of decorrelation. Thus from Figure 8.2 it may be determined that when the correlation coefficient between adjacent auxiliary antennas drops from 1.0, perfect correlation, to 0.9985, the PSK modem analyzed suffers approximately a 3 dB degradation in SNR performance. However, as the results of Figure 6.2 indicate, the addition of another auxiliary antenna will cause a very large improvement in processing gain and remove the SNR degradation for the same amount of decorrelation.

While a computer program has been developed to handle quite general three-dimensional antenna configurations and multiple jammers, there has not been time to generate results for this report. Also, the theory developed for error rate is valid for multiple jammers and antennas but insufficient time was available to calculate results for this report. The results obtained should be extended to include the effect of time and frequency selective fading. When the bandwidths become so large that system operation is impossible due to frequency selective fading, the system must be modified to include adaptive equalize s at the outputs of antennas.

SECTION II

SYSTEMS TO BE ANALYZED

Figure 2-1 shows a simplified block diagram of a fully adaptive array showing M elements and M complex weights connected to the antenna element outputs followed by a combiner. For simplicity, all frequency conversion and filtering operations are not shown. The combiner output c(t) is fed to the modem and to the adaptation circuitry. A reference signal r(t) is extracted from the modem and/or combiner output. An error signal e(t) is formed by subtracting the combiner output from the reference signal

$$e(t) = r(t) - c(t) \tag{2-1}$$

The error signal and the antenna element outputs are fed to a signal processor that implements an adaptation algorithm for the complex weights $p_1, \dots p_M$.

The transmitter nodes are labeled 0, 1, ... L with the signal s(t) being transmitted from node 0 and the jammers $j_1(t), \dots j_L(t)$ from nodes 1, 2, ... L, respectively. At the k^{th} antenna element, the received process is represented as the sum of three terms

$$w_k(t) = s_k(t) + i_k(t) + n_k(t) \tag{2-2}$$

a signal term $s_k(t)$, a jammer term $i_k(t)$, and an additive thermal noise term $n_k(t)$. The jammer term $i_k(t)$ may be represented as the sum of contributions from the L jammers to the k^{th} antenna element

$$i_k(t) = \sum_{m=1}^{L} i_{mk}(t) \tag{2-3}$$

where $i_{mk}(t)$ is the jamming signal received at the k^{th} antenna element from the m^{th} jammer.

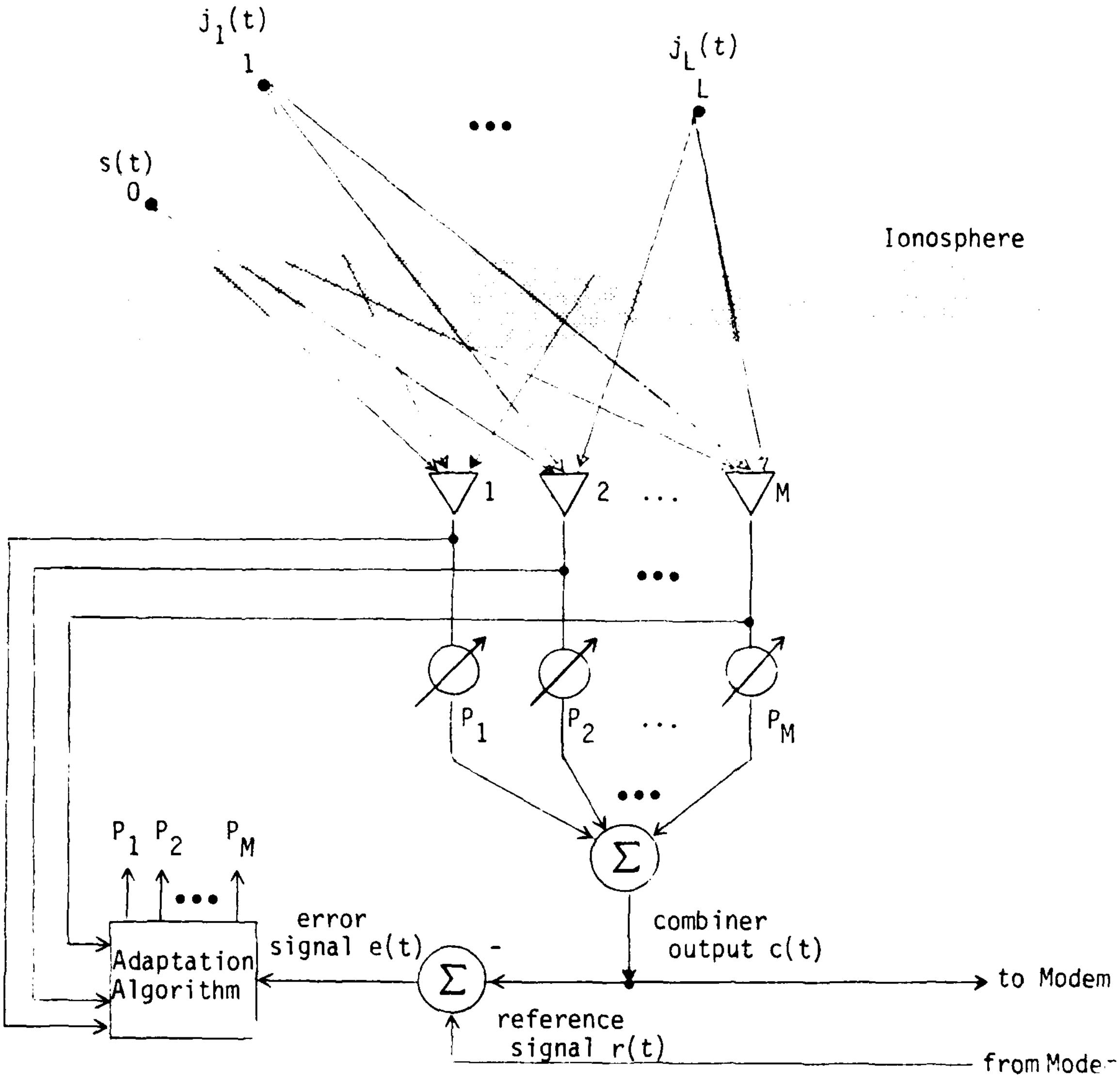

Figure 2-1 Simplified Diagram of Fully Adaptive Array

The signal term $s_k(t)$ may be expressed in the form

$$s_k(t) = \int s(t-\xi)\, h_{0k}(t,\xi)\, d\xi \qquad (2\text{-}4)$$

where $h_{0k}(t,\xi)$ is the time variant impulse response of the channel from node 0 to antenna element k. In general we represent the time-variant impulse response from node m to antenna element k as $h_{mk}(t,\xi)$. The corresponding time variant transfer function is

$$H_{mk}(f,t) = \int h_{mk}(t,\xi)\, e^{-j2\pi f\xi}\, d\xi \qquad (2\text{-}5)$$

In terms of the time variant transfer function

$$s_k(t) = \int S(f)\, H_{0k}(f,t)\, e^{j2\pi ft}\, df \qquad (2\text{-}6)$$

where S(f) is the spectrum of the transmitted signal.

The received jamming signal $i_{mk}(t)$ has the two equivalent expressions

$$i_{mk}(t) = \int j_m(t-\xi)\, h_{mk}(t,\xi)\, d\xi \qquad (2\text{-}7)$$

$$i_{mk}(t) = \int J_m(f)\, H_{mk}(f,t)\, e^{j2\pi ft}\, df \qquad (2\text{-}8)$$

where $J_m(f)$ is the spectrum of $j_m(t)$.

The combiner output is given by

$$c(t) = \sum_{k=1}^{M} p_k\, w_k(t) \qquad (2\text{-}9)$$

The system of Figure 2-1 uses an array of wide beam antenna elements or subarrays. An alternate approach is to form multiple orthogonal beams by a linear transformation (e.g., Butler matrix) and apply the adaptivity on the beam port outputs as shown in Figure 2-2. $Q \leq M$ beams are formed by separate linear transformations on the M subarray outputs. The Q beamport outputs are fed to adaptive complex weights and the adaptation circuitry is the same as in Figure 2-1.

Another system configuration of interest is the sidelobe canceller or power nuller shown in Figure 2-3. Under normal conditions the desired signal appears only in the main beam of the signal antenna and the jammer appears in the sidelobes. The auxiliary antennas perceive the jamming signals and by weighting the outputs of the auxiliary antennas and subtracting from the signal antenna output attempts to cancel the jamming signal. The system of Figure 2-3 can be cast in the framework of Figure 2-1 by identifying the signal antenna output as the reference signal. With ionospheric scintillation present, the transmitted signal may appear in the sidelobes and auxiliary antenna outputs and the jammer may appear in the signal antenna. For this reason we have shown paths from each transmitter to each antenna.

In this report we will confine our attention to the systems of Figures 2-1 and 2-3.

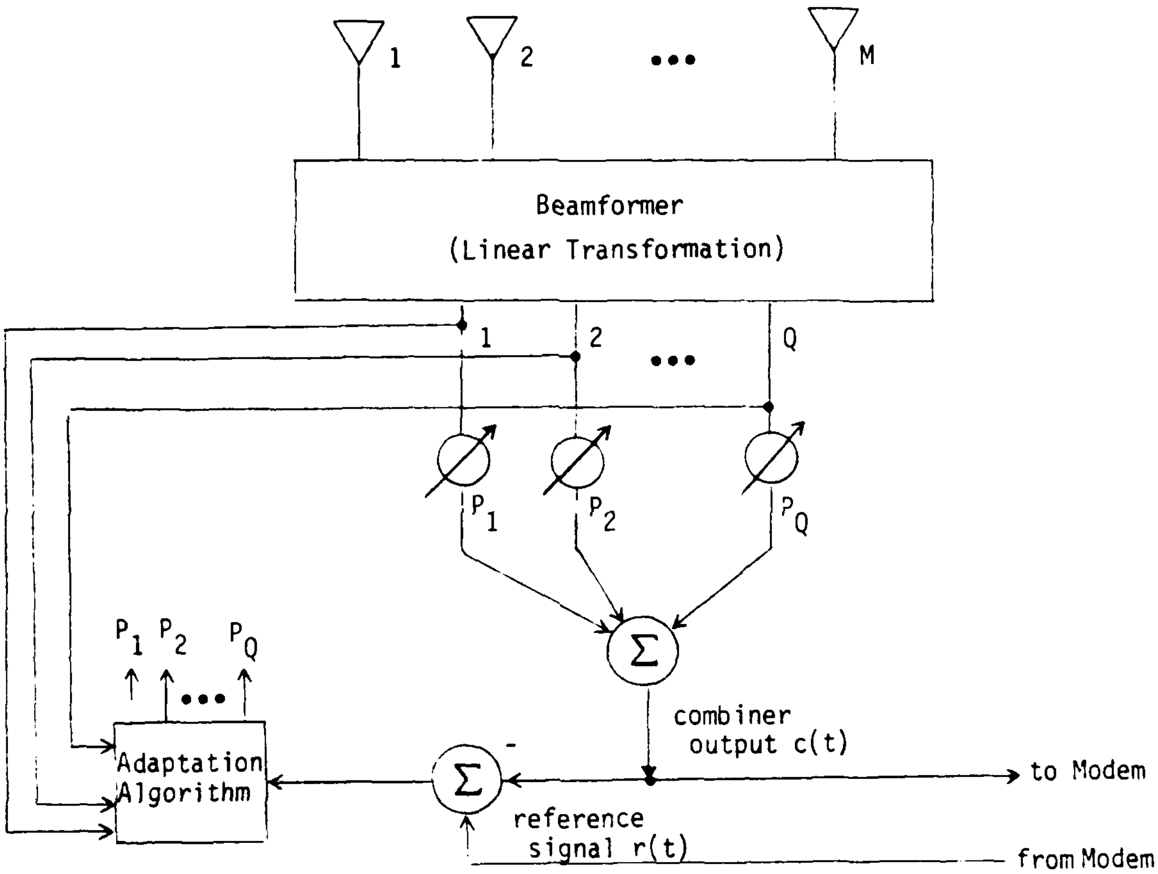

Figure 2-2 Simplified Diagram of Adaptive Beam Configuration

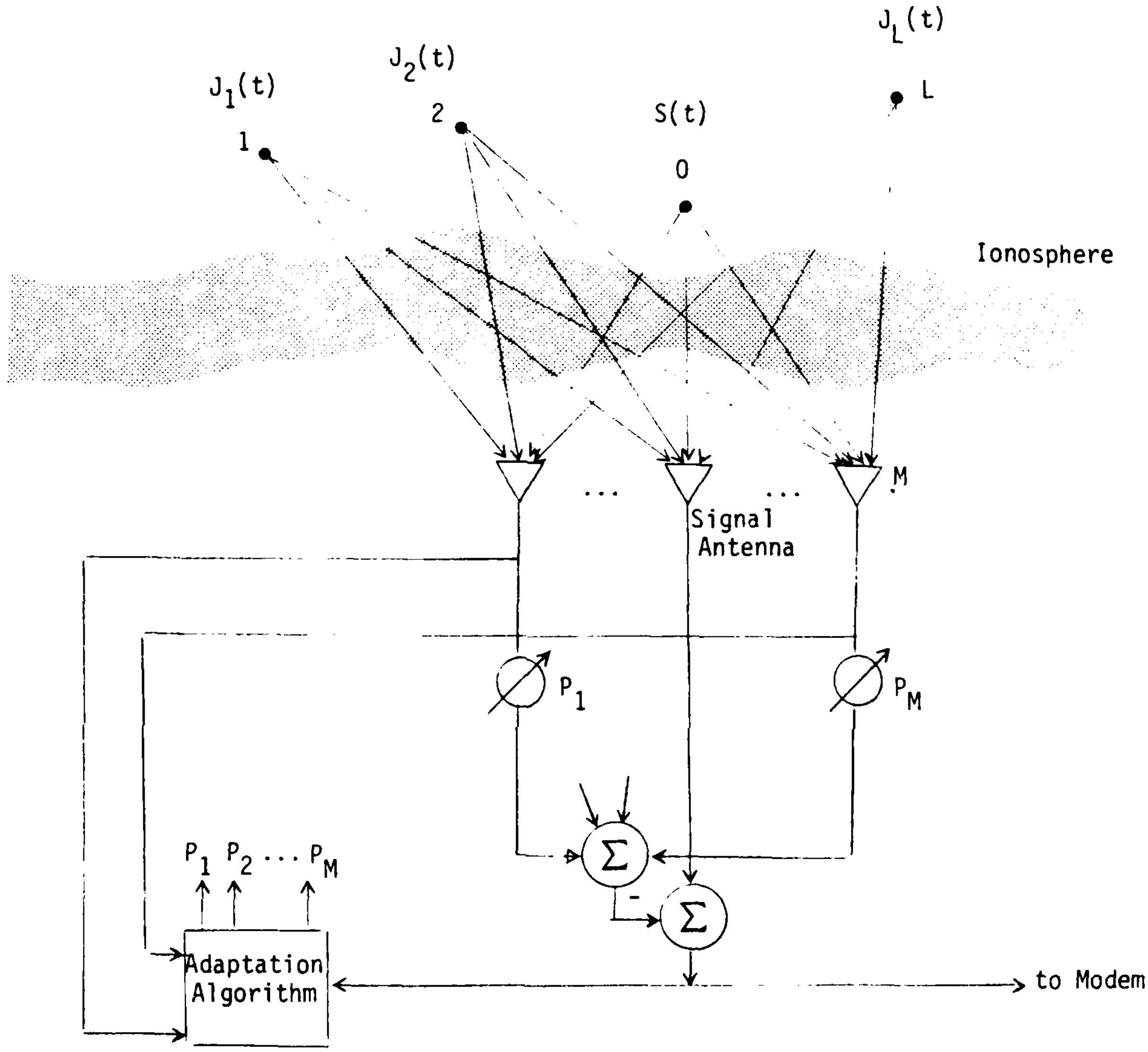

Figure 2-3 Simplified Diagram of Sidelobe Canceller System

SECTION III

OPTIMUM PERFORMANCE

In the performance analyses we shall have occasion to assume either complete adaptation to the fading or no adaptation to the fading. In the former case it is assumed in effect that the time constants of the adaptation circuits are sufficiently small for the system to follow the instantaneous fluctuations in the time variant transfer functions $H_{mk}(f,t)$. In the latter case the adaptive circuits average over the fading. (We neglect non-idealities in the adaptation process and circuitry.) In the former case the weights $\{p_k(t);\ k=1,2,...M\}$ are continually adjusted to minimize the mean squared error signal $\overline{|e(t)|^2}$ averaged over the transmitted signal and jammer statistics and the thermal noise fluctuations. The resultant mean squared error, weights, antenna pattern, SNR's, etc. will be time variable even with non-moving jammers and signal sources because of the fluctuations of the channels. In the latter case the system will adapt to changes in the channel statistics only.

It is well known (e.g., see [3.1]) that the solution for the weights which yield the minimum mean squared error is given by

$$\underline{P} = \underline{C}^{-1}\,\underline{R} \tag{3-1}$$

where

$$\underline{P}^T = \left[p_1,\ p_2,\ \ldots,\ p_M\right] \tag{3-2}$$

$$\underline{R}^T = \left[\overline{rw_1^*},\ \overline{rw_2^*},\ \ldots,\ \overline{rw_M^*}\right] \tag{3-3}$$

$$\underline{C} = \begin{bmatrix} \overline{|w_1|^2} & \overline{w_1^* w_2} & \cdots & \overline{w_1^* w_M} \\ \overline{w_2^* w_1} & \overline{|w_2|^2} & \cdots & \overline{w_2^* w_M} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{w_M^* w_1} & \overline{w_M^* w_2} & \cdots & \overline{|w_M|^2} \end{bmatrix} \tag{3-4}$$

in which T denotes matrix transpose and we have suppressed the time variable to simplify notation. It should be understood that the averages assume the channels "frozen".

The value of the minimum mean squared error is given by [3.1]

$$\overline{|e|^2} = \overline{|r|^2} - \underline{R}^H \underline{C}^{-1} \underline{R} \tag{3-5}$$

or, alternatively, from (3-1)

$$\overline{|e|^2} = \overline{|r|^2} - \underline{R}^H \underline{P} \tag{3-6}$$

where the symbol H denotes a combination of matrix transpose and complex conjugate operations ("Hermitian" transpose).

To formulate an expression for the antenna pattern, it is necessary to specify the geometry of the antenna subarray locations. For simplicity we assume a linear array with half wavelength spacing between subarray centers. In such a case the antenna gain pattern is expressed as

$$G(\phi) = G_s(\phi) \left| \sum_{k=1}^{M} \rho_k \, e^{j\pi(k-1)\sin\phi} \right|^2 \tag{3-7}$$

where $G_s(\phi)$ is the antenna gain pattern for a single subarray and ϕ is the angle measured from boresight.

We may formulate an expression for the output SNR, ρ. This discussion of SNR is for the fully adaptive array and must be modified for the SLC (sidelobe canceller) configuration. See Section 5.

$$\rho = \frac{\overline{\left| \sum_{k=1}^{M} p_k \; s_k \right|^2}}{\overline{\left| \sum_{k=1}^{M} p_k (i_k + n_k) \right|^2}} = \frac{\underline{P}^H \; \underline{S} \; \underline{P}}{\underline{P}^H \; (N\underline{I} + \underline{J}) \; \underline{P}} \tag{3-8}$$

where $\underline{I}$ is the identity matrix,

$$N = \overline{|n_k|^2} \tag{3-9}$$

$$\underline{S} = \begin{bmatrix} \overline{|s_1|^2} & \overline{s_1^* s_2} & \cdots & \overline{s_1^* s_M} \\ \overline{s_2^* s_1} & \overline{|s_2|^2} & \cdots & \overline{s_2^* s_M} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{s_M^* s_1} & \overline{s_M^* s_2} & \cdots & \overline{|s_M|^2} \end{bmatrix} \tag{3-10}$$

$$\underline{J} = \begin{bmatrix} \overline{|i_1|^2} & \overline{i_1^* i_2} & \cdots & \overline{i_1^* i_M} \\ \overline{i_2^* i_1} & \overline{|i_2|^2} & \cdots & \overline{i_2^* i_M} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{i_M^* i_1} & \overline{i_M^* i_2} & \cdots & \overline{|i_M|^2} \end{bmatrix} \tag{3-11}$$

and it has been assumed that the thermal noises in the different antenna elements are statistically independent and of the same level.

Since

$$\underline{C} = \underline{S} + \underline{J} + N\underline{I} \tag{3-12}$$

(3-8) may be expressed in the alternate form

$$\rho = \frac{\underline{P}^H \, \underline{C} \, \underline{P}}{\underline{P}^H \, (N\underline{I} + \underline{J}) \, \underline{P}} - 1 \tag{3-13}$$

and from (3-1),(3-6) we obtain the further equivalent expressions

$$\rho = \frac{\underline{R}^H \, \underline{P}}{\underline{P}^H \, (N\underline{I} + \underline{J}) \, \underline{P}} - 1 \tag{3-14}$$

$$\rho = \frac{\overline{|r|^2} - \overline{|e|^2}}{\underline{P}^H \, (N\underline{I} + \underline{J}) \, \underline{P}} - 1 \tag{3-15}$$

We now consider the evaluation of the averages required to determine $\underline{P}$, $\overline{|e|^2}$, ρ, and $G(\phi)$. In the evaluation we shall assume that the signal and jamming processes are statistically stationary. Thus [3.2]

$$\overline{S^*(f) \; S(\ell)} = P(f) \; \delta(f-\ell) \tag{3-16}$$

$$\overline{J_m^*(f) J_m(\ell)} = P_m(f) \; \delta(f-\ell) \tag{3-17}$$

where $P(f)$, $P_m(f)$ are the power spectra of the transmitted signal and the m^{th} transmitted jamming signal and $\delta(\cdot)$ is the unit impulse function. We shall also assume that the various jamming signals are statistically independent to simplify notation.

The received signal crosscorrelations are given by

$$\overline{s_k s_q^*} = \iint \overline{S^*(f)S(\ell)}\ H_{0k}{}^*(f,t)\ H_{0q}(\ell,t)\ e^{j2\pi(\ell-f)t}\ dfd\ell \qquad (3\text{-}19)$$

Using (3-16)

$$\overline{s_k s_q^*} = \int P(f)\ H_{0k}{}^*(f,t)\ H_{0q}(f,t)\ df \qquad (3\text{-}20)$$

The received jamming signal crosscorrelations are given by

$$\overline{i_k i_q^*} = \sum \overline{i_{mk}\ i_{mq}{}^*}$$

$$= \sum_k \iint \overline{J_m{}^*(f)\ J_m(\ell)}\ H_{mk}{}^*(f,t)\ H_{mq}(\ell,t)\ e^{j2\pi(\ell-f)t}\ dfd\ell \qquad (3\text{-}21)$$

Using (3-18)

$$\overline{i_k i_q^*} = \sum_k \int P_m(f)\ H_{mk}{}^*(f,t)\ H_{mq}(f,t)\ df \qquad (3\text{-}22)$$

We see that in general it is necessary to specify the joint statistical properties of the various transfer functions to evaluate the statistical properties of the optimum solution to the adaptive array problem. In the case of strong ionospheric scintillations the channel transfer functions become complex Gaussian processes. Then their joint statistical properties can be determined from knowledge of the correlation functions

$$\overline{H_{nk}{}^*(f,t)\ H_{mq}(\ell,s)} = R_{nmkq}(f,\ell;t,s) \quad ; \begin{cases} m,n = 0,1,\ldots,L \\ k,q = 1,2,\ldots,M \end{cases} \qquad (3\text{-}23)$$

This correlation function is more general than the mutual coherence functions normally considered, in that we are asking for the time, frequency, and spatial coherence functions for received signals corresponding to transmission of carriers from two different locations in space rather than one.

SECTION IV

TWO-ELEMENT EXAMPLE FOR FULLY ADAPTIVE ARRAY; NON-FREQUENCY SELECTIVE CASE

In this section we apply the results of the previous section to the simplest non-trivial example, the case of two omni-directional antenna elements separated by a half-wavelength as shown in Figure 4-1. Such a configuration is able to discriminate against a single jammer. We confine our attention in this section to signaling elements that are sufficiently narrowband that no frequency selective distortion is caused by the disturbed trans-ionospheric channel.

4.1 PERFORMANCE WITHOUT CHANNEL DISTURBANCE

It is instructive to compare the performance before and after the onset of the channel disturbance. In the absence of a channel disturbance, it is sufficient to model the quiet or non-disturbed channel by a simple phase shift for the case of narrowband systems, as assumed here. With the desired transmitted signal on boresight we may use

$$H_{01}(f,t) = H_{02}(f,t) = 1 \tag{4-1}$$

$$H_{11}(f,t) = 1 \tag{4-2}$$

$$H_{12}(f,t) = e^{j\pi \sin\theta} \tag{4-3}$$

To simplify notation we have neglected path losses. This results in no loss of generality since our answers are expressed in terms of <u>received</u> power levels.

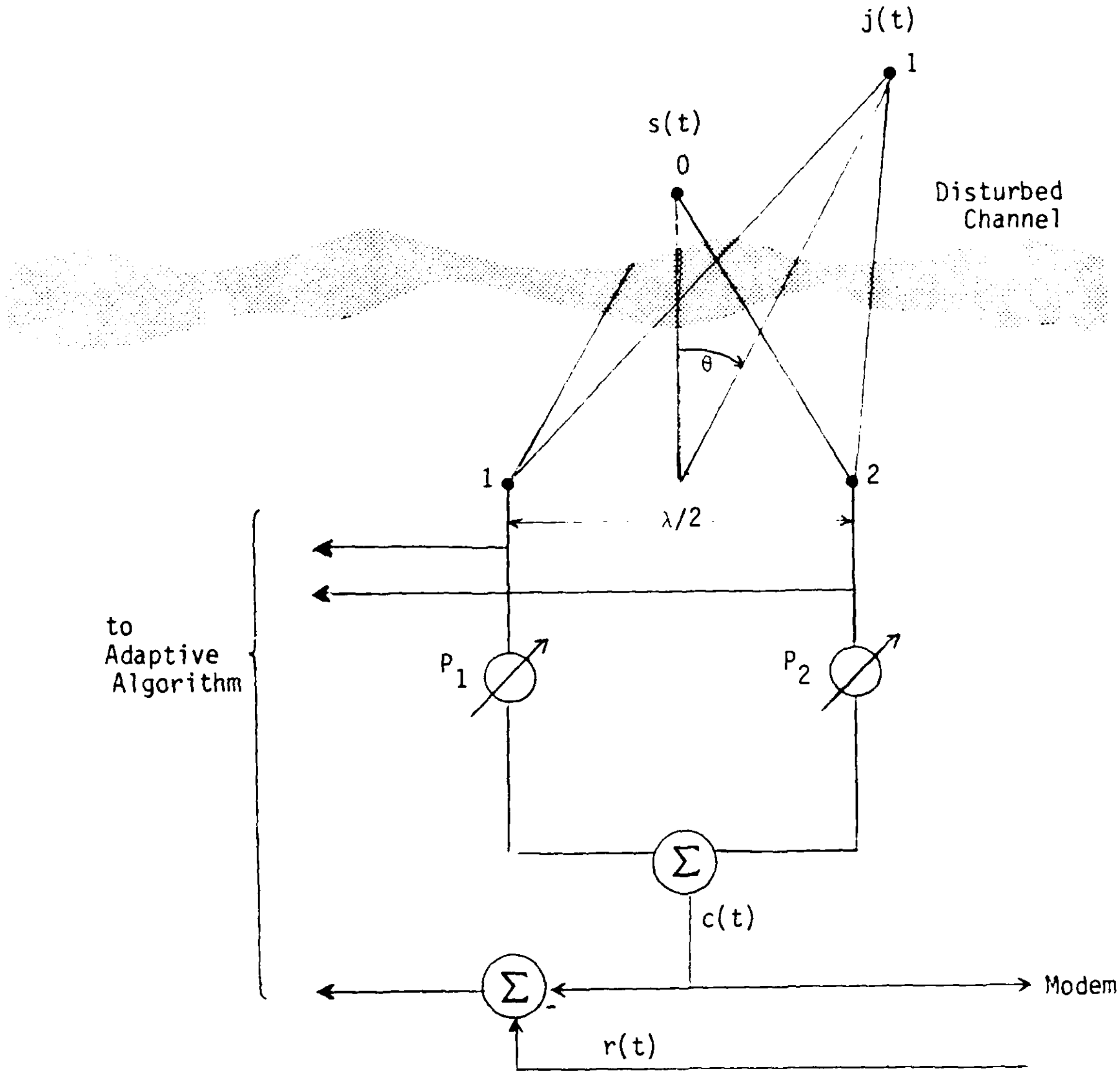

Figure 4-1 Two-Element Example of Fully Adaptive Array

With the degenerate transfer functions (4-1) through (4-3)

$$w_1(t) = s(t) + j(t) + n_1(t) \tag{4-4}$$

$$w_2(t) = s(t) + j(t)\, e^{j\pi \sin\theta} + n_2(t) \tag{4-5}$$

and from (3-4)

$$C = \begin{bmatrix} S+J+N & S+Je^{j\pi\sin\theta} \\ S+Je^{-j\pi\sin\theta} & S+J+N \end{bmatrix} \tag{4-6}$$

where

$$S = \overline{|s(t)|^2} \tag{4-7}$$

$$J = \overline{|j(t)|^2} \tag{4-8}$$

The reference signal r(t) is taken equal to the signal s(t),

$$r(t) = s(t) \tag{4-9}$$

so that [see (3-3)]

$$\underline{R}^T = \begin{bmatrix} S & , S \end{bmatrix} \tag{4-10}$$

Then, using (3-1) we find that the optimum weights are

$$C = \begin{bmatrix} S+J+N & S+Je^{j\pi\sin\theta} \\ S+Je^{-j\pi\sin\theta} & S+J+N \end{bmatrix}^{-1} \begin{bmatrix} S \\ S \end{bmatrix} \tag{4-11}$$

Upon carrying out the algebra, we find that

$$\underline{P} = \beta\left(\frac{S}{N}\right)\begin{bmatrix} \frac{J}{N}\left(1 - e^{j\pi\sin\theta}\right) + 1 \\ \frac{J}{N}\left(1 - e^{-j\pi\sin\theta}\right) + 1 \end{bmatrix} \tag{4-12}$$

where

$$\beta = \frac{1}{4\left(\frac{J}{N}\right)\left(\frac{S}{N}\right)\sin^2\left(\frac{\pi}{2}\sin\theta\right) + 2\left(\frac{J}{N} + \frac{S}{N}\right) + 1} \tag{4-13}$$

The minimum mean squared error is obtained by using (4-12) and (4-10) in (3-6). Carrying out the operations and normalizing the error to the strength of the desired signal,

$$\frac{\overline{|e|^2}}{S} = \beta\left(2\left(\frac{J}{N}\right) + 1\right) \tag{4-14}$$

For a large jamming signal ($J >> S$) and θ not near zero

$$\frac{\overline{|e|^2}}{S} \rightarrow \frac{N}{2S\sin^2\left(\frac{\pi}{2}\sin\theta\right)} \tag{4-15}$$

Turning to the calculation of output SNR (3-8) we note that

$$\underline{P}^H\,\underline{S}\,\underline{P} = \beta^2\left(\frac{S}{N}\right)^2\begin{bmatrix} \frac{J}{N}(1-e^{j\psi})+1, & \frac{J}{N}(1-e^{-j\psi})+1 \end{bmatrix}\begin{bmatrix} S & S \\ S & S \end{bmatrix}\begin{bmatrix} \frac{J}{N}(1-e^{-j\psi})+1 \\ \frac{J}{N}(1-e^{j\psi})+1 \end{bmatrix} \tag{4-16}$$

where

$$\psi = \pi \sin \theta \tag{4-17}$$

Upon carrying out the calculations we find that

$$\underline{P}^H \underline{S}\, \underline{P} = \beta^2 S\left(\frac{S}{N}\right)^2 4 \left[2\left(\frac{J}{N}\right) \sin^2\left(\frac{\pi}{2} \sin\theta\right) + 1\right]^2 \tag{4-18}$$

For the output jamming and noise power we have

$\underline{P}^H(N\underline{I}+\underline{J})\underline{P} =$

$$\beta^2\left(\frac{S}{N}\right)^2 \left[\frac{J}{N}(1-e^{j\psi})+1, \frac{J}{N}(1-e^{-j\psi})+1\right] \begin{bmatrix} J+N & J e^{-j\psi} \\ Je^{-j\psi} & J+N \end{bmatrix} \begin{bmatrix} \frac{J}{N}(1-e^{-j\psi})+1 \\ \frac{J}{N}(1-e^{j\psi})+1 \end{bmatrix} \tag{4-19}$$

Again, carrying out the operations

$$\underline{P}^H(N\underline{I}+\underline{J})\underline{P} = \beta^2 N\left(\frac{S}{N}\right)^2 \left(2\frac{J}{N}+1\right) 2\left(2\frac{J}{N}\sin^2\left(\frac{\pi}{2}\sin\theta\right) + 1\right) \tag{4-20}$$

which yields the output SNR

$$\rho = \frac{\underline{P}^H \underline{S}\, \underline{P}}{\underline{P}^H(N\underline{I}+\underline{J})P} = \frac{2S}{N} \left(\frac{2\frac{J}{N} \sin^2\left(\frac{\pi}{2}\sin^2\theta\right)+1}{2\frac{J}{N} + 1}\right) \tag{4-21}$$

We note the limiting cases

$$\rho \to \frac{2S}{N} \qquad \text{for} \quad \theta \to \frac{\pi}{2} \tag{4-22}$$

$$\rho \to \frac{2S}{2J+N} \qquad \text{for} \quad \theta \to 0 \tag{4-23}$$

$$\rho \to \frac{2S}{N} \sin^2\left(\frac{\pi}{2}\sin\theta\right) \qquad \text{for } \frac{J}{N} \sin^2\left(\frac{\pi}{2}\sin\theta\right) >> 1 \tag{4-24}$$

For $\theta \to \pi/2$ the jammer is completely nulled by the same weights which maximize the desired signal level, while for $\theta \to 0$ the jammer is located on boresight (as the desired transmitter) and no jammer suppression is possible.

It is intuitively clear that when the reference signal equals the desired received signal s(t), an appropriate measure of SNR is provided by the ratio of the signal strength S of the reference signal to the mean squared departure of the output signal from the reference signal, i.e., the ratio $S/\overline{|e|^2}$. From (4-13) and (4-14) we find that

$$\frac{S}{\overline{|e|^2}} = \frac{4(\frac{J}{N})(\frac{S}{N})\sin^2(\frac{\pi}{2}\sin\theta) + 2(\frac{J}{N} + \frac{S}{N}) + 1}{2\,(\frac{J}{N}+1)} \tag{4-25}$$

Using (4-21) in (4-24) we find that

$$\frac{S}{\overline{|e|^2}} = \rho + 1 \tag{4-26}$$

The expression for the antenna pattern $G(\phi)$ is given by (3-7). Using (4-12) in (3-7) with $G_S(\phi) = 1$,

$$G(\phi) = \beta^2(\frac{S}{N})^2 \left| (\frac{J}{N})(1-e^{j\pi\sin\theta})+1 + \left[(\frac{J}{N})(1-e^{-j\pi\sin\theta})+1\right] e^{j\pi\sin\phi} \right|^2 \tag{4-27}$$

As we have mentioned, when $\theta = \pi/2$ the jammer is completely cancelled and the antenna pattern is set for maximum gain at $\phi = 0$. Collecting terms in (4-27) and normalizing to G(0) for $\theta = \pi/2$,

$$\frac{G(\phi)}{[G(0)]_{\theta=\pi/2}} = \left| \frac{(\frac{J}{N} + 1)\cos(\frac{\pi}{2}\sin\phi) - \frac{J}{N}\cos\pi(\sin\theta - \frac{1}{2}\sin\phi)}{2\frac{J}{N} + 1} \right|^2 \qquad (4\text{-}28)$$

4.2 FLAT FADING WITH SPATIAL DECORRELATION: ARRAY ADAPTIVE TO FADING

The first case we shall examine with channel distortion is that of sufficiently narrowband transmission that essentially flat fading results. However it is assumed that the antennas are far enough apart that there is some decorrelation between the perceived jammer and signal channel fluctuations at the two antennas. We also assume in this section that the time constants of the adaptive circuitry are small enough to allow adaptation to the fading.

With the above assumptions we may express the channel transfer functions in the degenerate forms

$$H_{01}(f,t) = g_{01}(t) \tag{4-29}$$

$$H_{02}(f,t) = g_{02}(t) \tag{4-30}$$

$$H_{11}(f,t) = g_{11}(t) \tag{4-31}$$

$$H_{12}(f,t) = g_{12}(t)\, e^{j\pi \sin\theta} \tag{4-32}$$

in which $g_{mn}(t)$ becomes a complex Gaussian process in the limit of strong scintillation.

The received processes are now given by

$$w_1(t) = g_{01}(t)s(t) + g_{11}(t)j(t) + n_1(t) \tag{4-33}$$

$$w_2(t) = g_{02}(t)s(t) + g_{02}(t)j(t)\, e^{j\pi \sin\theta} + n_2(t) \tag{4-34}$$

and the crosscorrelation matrix by [see (3-4)]

$$\underline{C} = \begin{bmatrix} |g_{01}|^2 S + |g_{11}|^2 J + N & g_{01}^* g_{02} S + g_{11}^* g_{12} J e^{j\pi \sin\theta} \\ g_{01} g_{02}^* S + g_{11} g_{12}^* J e^{-j\pi \sin\theta} & |g_{02}|^2 S + |g_{12}|^2 J + N \end{bmatrix} \tag{4-35}$$

where we have temporarily omitted the argument (t) from the complex channel gains to simplify notation.

The reference signal-input process crosscorrelation vector takes the form [see (3-3) and use (4-9)],

$$\underline{R}^T = \begin{bmatrix} g_{01}^* S \; , \; g_{02}^* S \end{bmatrix} \tag{4-36}$$

To solve for the adaptive weights we require $\underline{C}^{-1}$. This inverse is found to be

$$\underline{C}^{-1} = \frac{1}{\Delta} \begin{bmatrix} |g_{02}|^2 S + |g_{12}|^2 J + N & -(g_{01}^* g_{02} S + g_{11}^* g_{12} J e^{j\pi \sin\theta}) \\ -(g_{01} g_{02}^* S + g_{11} g_{12}^* J e^{-j\pi \sin\theta}) & |g_{01}|^2 S + |g_{11}|^2 J + N \end{bmatrix} \tag{4-37}$$

where the determinant

$$\Delta = (|g_{01}|^2 S + |g_{12}|^2 J + N)(|g_{02}|^2 S + |g_{12}|^2 J + N) - \left| g_{01}^* g_{02} S + g_{11}^* g_{12} J e^{j\pi \sin\theta} \right|^2 \tag{4-38}$$

Upon carrying out the algebra in (4-36) we find

$$\Delta = SJ\,|g_{01}|^2\,|g_{12}|^2 \left| 1 - e^{-j\pi\sin\theta}\left(\frac{g_{11}}{g_{12}}\right)\left(\frac{g_{02}}{g_{01}}\right)\right|^2 +$$

$$N\left(\left[|g_{01}|^2 + |g_{02}|^2\right] S + \left[|g_{11}|^2 + |g_{12}|^2\right] J\right) + N^2 \quad (4\text{-}39)$$

We compute the weight vector via the matrix operation (3-1)

$$\underline{P} = \underline{C}^{-1}\underline{R} = \frac{1}{\Delta}\begin{bmatrix} \left(g^*_{01}\,|g_{12}|^2 - g^*_{11}g_{12}g^*_{02}\,e^{j\pi\sin\theta}\right)SJ + g^*_{01}\,SN \\ \left(g^*_{02}\,|g_{11}|^2 - g_{11}g^*_{12}g^*_{01}\,e^{-j\pi\sin\theta}\right)SJ + g^*_{02}\,SN \end{bmatrix} \quad (4\text{-}40)$$

The mean squared error is obtained from (3-6)

$$\overline{|e|^2} = S - \underline{R}^H\,\underline{P} =$$

$$S - \frac{1}{\Delta}\left[g_{01}S, g_{02}S\right]\begin{bmatrix} \left(g^*_{01}\,|g_{12}|^2 - g_{11}g^*_{12}g^*_{02}\,e^{j\pi\sin\theta}\right)SJ + g^*_{01}\,SN \\ \left(g^*_{02}\,|g_{11}|^2 - g_{11}g^*_{12}g^*_{01}\,e^{-j\pi\sin\theta}\right)SJ + g^*_{02}\,SN \end{bmatrix}$$

(4-41)

The result is

$$\frac{\overline{|e|^2}}{S} = \frac{\frac{J}{N}\left(|g_{11}|^2 + |g_{12}|^2\right) + 1}{\left(\frac{J}{N}\right)\left(\frac{S}{N}\right)|g_{01}|^2\,|g_{12}|^2\left|1 - e^{-j\pi\sin\theta}\left(\frac{g_{11}}{g_{12}}\right)\left(\frac{g_{02}}{g_{01}}\right)\right|^2 + \frac{S}{N}\left(|g_{01}|^2 + |g_{02}|^2\right) + \frac{J}{N}\left(|g_{11}|^2 + |g_{12}|^2\right) + 1} \quad (4\text{-}42)$$

It is worth pausing to examine the expression (4-41). Assuming the usual case that the jammer is much stronger than the signal and the signal much stronger than the noise, i.e.,

$$J \gg S \gg N \tag{4-43}$$

with the average strengths of the channel gains normalized to unity, i.e.,

$$\overline{|g_{11}|^2} = \overline{|g_{12}|^2} = \overline{|g_{01}|^2} = \overline{|g_{02}|^2} = 1 \tag{4-44}$$

then, except where the term

$$A(\theta,t) = \left| 1 - e^{-j\pi\sin\theta}\left(\frac{g_{11}}{g_{12}}\right)\left(\frac{g_{02}}{g_{01}}\right) \right|^2 \tag{4-45}$$

is very much smaller than unity, we have the approximation

$$\frac{\overline{|e|^2}}{S} \approx \left(\frac{|g_{11}|^2 + |g_{12}|^2}{|g_{12}|^2}\right)\left(\frac{N}{S\,|g_{10}|^2}\right)\frac{1}{A(\theta,t)} \tag{4-46}$$

Let us first examine the form this expression takes when the spatial decorrelation vanishes, i.e., when

$$g_{11} = g_{12} \tag{4-47}$$

$$g_{01} = g_{02} \tag{4-48}$$

Then the first factor becomes the numeric 2 and

$$A(\theta,t) = 4\sin^2\left(\frac{\pi}{2}\sin\theta\right) \tag{4-49}$$

yielding

$$\frac{\overline{|e|^2}}{S} \approx (2)\left(\frac{N}{S\,|g_{01}|^2}\right)\left(\frac{1}{4\sin^2\left(\frac{\pi}{2}\sin\theta\right)}\right) \tag{4-50}$$

We see that the dependence of the mean squared error on the jammer angular position θ is the same as the non-fading channel, but the level of the mean squared error is strongly influenced by the instantaneous strength of the signal channel, i.e., of the received signal level. In a deep signal fade the mean squared error will become much larger, as one would expect. However this is the normal impact of a fading signal and must be counteracted by modem design. At any given signal level, however, the adaptive array is doing its job of suppressing the jammer via the third factor in (4-50).

Returning to (4-46) and assuming that there is a small amount of spatial decorrelation, the first factor in (4-46) will reside close to the numeric two on the average. The second factor is the same as in the case of zero spatial decorrelation. Thus it is primarily the third factor which brings in the effect of jammer angular position on mean squared error and we must examine this term to assess the impact of spatial decorrelation on the dependence of jammer nulling on jammer angular position.

We consider now the calculation of output SNR (3-8). The relevant matrices are

$$\underline{S} = S \begin{bmatrix} |g_{01}|^2 & g_{01}^* g_{02} \\ g_{01} g_{02}^* & |g_{02}|^2 \end{bmatrix} \qquad (4\text{-}51)$$

$$N\underline{I} + \underline{J} = \begin{bmatrix} |g_{11}|^2 J + N & g_{11}^* g_{12} J \, e^{j\pi \sin\theta} \\ g_{11} g_{12}^* J \, e^{-j\pi \sin\theta} & |g_{12}|^2 J + N \end{bmatrix} \qquad (4\text{-}52)$$

After laborious calculation of

$$\rho = \frac{\underline{P}^H \underline{S} \underline{P}}{\underline{P}^H (N\underline{I}+\underline{J}) \underline{P}^H} \tag{4-53}$$

it was found, again that

$$\frac{S}{\overline{|e|^2}} = \rho + 1 \tag{4-54}$$

Apparently the relationship (4-54) is valid for the non-frequency selective narrowband channel case. It is not known under what more general conditions (4-54) may be valid.

Using (4-46) we find that

$$\rho = \left(\frac{S}{N}\right) \frac{\left(\frac{J}{N}\right) A(\theta,t) |g_{01}|^2 |g_{12}|^2 + |g_{01}|^2 + |g_{02}|^2}{\left(\frac{J}{N}\right) \left(|g_{11}|^2 + |g_{12}|^2\right) + 1} \tag{4-55}$$

In the absence of jamming

$$\rho = \left(\frac{S}{N}\right) \left(|g_{01}|^2 + |g_{02}|^2\right) \tag{4-56}$$

Thus the adaptive array operation produces dual diversity with square law combining in the absence of jamming.

For strong jamming

$$\rho \approx \frac{S}{N} |g_{01}|^2 \left(\frac{|g_{12}|^2}{|g_{11}|^2 + |g_{12}|^2}\right) A(\theta,t) \quad ; \quad \frac{J}{N} A(\theta,t) >> 1 \tag{4-57}$$

Using Eq.(3-7) and the adaptive weights (4-40) we find that the antenna gain pattern

$$G(\phi) \quad \left|\frac{J}{N}\left(g_{12}^{*} - e^{-j\pi(\sin\phi - \sin\theta)} g_{11}^{*}\right) g_{01} g_{12}\ A(\theta,t) + \left(g_{01} + g_{02}\ e^{-j\pi\sin\phi}\right)\right|^{2} \tag{4-58}$$

The ratio of the gain towards the signal location $\phi=0$ multiplied by the signal power to the gain towards the jammer location $\phi=\theta$ multiplied by the jammer power is sometimes of interest. This is just

$$\frac{SG(0)}{JG(\theta)} = \frac{S}{N} \frac{\left|\frac{J}{N}\left(g_{12}^{*} - e^{-j\pi\sin\theta} g_{11}^{*}\right)\ g_{01} g_{12} A(\theta,t) + \left(g_{01} + g_{02}\right)\right|^{2}}{\left|\frac{J}{N}\left(g_{12}^{*} - g_{11}^{*}\right)\ g_{01} g_{12} A(\theta,t) + \left(g_{01} + g_{02}\ e^{-j\pi\sin\theta}\right)\right|^{2}} \tag{4-59}$$

Assuming no spatial decorrelation but still scintillation

$$g_{12} = g_{11} \tag{4-60}$$

$$g_{01} = g_{02} \tag{4-61}$$

and (4-59) simplifies to

$$\frac{G(0)}{G(\theta)} = \frac{S}{J} \left| \frac{2\frac{J}{N} \quad |g_{11}|^{2} \sin^{2}\left(\pi \sin\frac{\theta}{2}\right) + 1}{\cos\left(\pi \sin\frac{\theta}{2}\right)} \right|^{2} \tag{4-62}$$

which differs from the non-fading ideal channel case only in that the factor $|g_{11}|^{2}$ accounting for the jammer power fluctuation is present.

4.3 FLAT FADING WITH SPATIAL DECORRELATION: ARRAY NON-ADAPTIVE TO FADING

In the previous section we studied the performance of the two-element adaptive array assuming that the adaptation circuits can adapt to the assumed flat fading. Here we consider the other extreme: adaptation circuit time constant larger than the fading time constant. In such a case the circuits average over both the input processes and the fading fluctuations. The crosscorrelation matrix $\underline{C}$ (4-35) and crosscorrelation vector R^T (4-36) must then be changed to

$$\underline{C} = \begin{bmatrix} \overline{|g_{01}|^2}S + \overline{|g_{11}|^2}J + N & \overline{g_{01}^* g_{02}}S + \overline{g_{11}^* g_{12}}Je^{j\pi\sin\theta} \\ \overline{g_{01} g_{02}^*}S + \overline{g_{11} g_{12}^*}Je^{-j\pi\sin\theta} & \overline{|g_{02}|^2}S + \overline{|g_{12}|^2}J + N \end{bmatrix} \tag{4-63}$$

$$\underline{R}^T = \begin{bmatrix} \overline{g_{01}^*}S\ , & \overline{g_{02}^*}S \end{bmatrix} \tag{4-64}$$

where we have assumed $r(t) = s(t)$.

We note that if the fading is sufficiently severe that

$$\overline{g_{01}} = \overline{g_{02}} = 0 \tag{4-65}$$

as in the case of complex Gaussian fading, R^T will vanish and the optimum weights will go to zero, shutting off the array. This phenomenon does not occur in the case of a sidelobe canceller as will be discussed in the following section.

We normalize the channel strengths and define correlation coefficients and mean as follows

$$\overline{|g_{01}|^2} = \overline{|g_{02}|^2} = \overline{|g_{11}|^2} = \overline{|g_{12}|^2} = 1 \qquad (4\text{-}66)$$

$$\rho_0 \equiv \frac{\overline{g_{01}^{*} g_{02}}}{\sqrt{\overline{|g_{11}|^2}\quad\overline{|g_{02}|^2}}} \qquad (4\text{-}67)$$

$$\rho_1 = \frac{\overline{g_{11}^{*} g_{12}}}{\sqrt{\overline{|g_{11}|^2}\quad\overline{|g_{02}|^2}}} \qquad (4\text{-}68)$$

$$m_0 = \overline{g_{01}^{*}} = \overline{g_{02}^{*}} \qquad (4\text{-}69)$$

With these definitions

$$\underline{C} = \begin{bmatrix} S + J + N & \rho_0 S + \rho_1 J\, e^{j\pi\sin\theta} \\ \rho_0^{*} S + \rho_1^{*} J\, e^{-j\pi\sin\theta} & S + J + N \end{bmatrix} \qquad (4\text{-}70)$$

$$\underline{R}^T = \begin{bmatrix} mS & , mS \end{bmatrix} \qquad (4\text{-}71)$$

The inverse correlation matrix is given by

$$\underline{C}^{-1} = \frac{1}{\Delta} \begin{bmatrix} S + J + N & -\left(\rho_0 S + \rho_1 J\, e^{j\pi\sin\theta}\right) \\ -\left(\rho_0^{*} S + \rho_1^{*} J e^{-j\pi\sin\theta}\right) & S + J + N \end{bmatrix} \qquad (4\text{-}72)$$

where

$$\Delta = S^2(1 - |\rho_0|^2) + J^2(1 - |\rho_1|^2) + 2SJ(1 - \mathrm{Re}\{\rho_0^*\rho_1 e^{j\psi}\})$$
$$+ 2(S+J)N + N^2 \qquad (4\text{-}73)$$

Using Eq.(3-1) the optimum weights are given by

$$P = \frac{mSN}{\Delta}\begin{bmatrix} \frac{S}{N}(1-\rho_0) + \frac{J}{N}(1-\rho_1\, e^{j\pi\sin\theta}) + 1 \\ \frac{S}{N}(1-\rho_0^*) + \frac{J}{N}(1-\rho_1^*\, e^{-j\pi\sin\theta}) + 1 \end{bmatrix} \qquad (4\text{-}74)$$

Note that when there is no spatial decorrelation, i.e.,

$$\rho_0 = \rho_1 = 1 \qquad (4\text{-}75)$$

the optimum weights differ from the non-fading case only by the factor m. In fact it is seen that when (4-75) applies, the channel complex gains may be redefined as part of the corresponding transmitted processes s(t) and j(t). Consequently, antenna patterns and SNR's will then be the same as in the non-fading case. Mean squared errors will differ, however, because of the reduction in weights by the factor m.

If the spatial decorrelations become so severe that

$$\rho_1 = \rho_0 = 0 \qquad (4\text{-}76)$$

then (4-74) shows that the adaptive weights become equal in value. This will produce an antenna pattern optimized for the signal in the absence of jamming. Thus the jammer nulling capability has been completely destroyed.

Consider another special case. Suppose that there is no spatial decorrelation for the signal, that the magnitude of the jammer spatial decorrelation is close to unity but there is a non-zero angle, i.e.,

$$\rho_0 = 1 \tag{4-77}$$

$$|\rho_1| = 1 \tag{4-78}$$

$$\measuredangle\rho_1 = \gamma \tag{4-79}$$

then we see that the jamming is evident (apart from the factor m) only in a replacement

$$-\pi \sin\theta \rightarrow \gamma - \pi \sin\theta \tag{4-80}$$

i.e., from the point of view of the adaptive array the angular position of the jammer has been changed to a new angle ψ, where

$$\pi \sin\psi = \pi \sin\theta - \gamma \tag{4-81}$$

That is to say the null will be steered in error by the angle $\psi-\theta$.

The mean squared error is given by Eq.(3-6)

$$\overline{|e|^2} = S - 2\,\frac{|m|^2\, S^2 N}{\Delta}\left[\frac{S}{N}\,(1 - \mathrm{Re}\{\rho_0\}) + \frac{J}{N}\,(1 - \mathrm{Re}\{\rho_1 e^{j\pi \sin\theta}\}) + 1\right] \tag{4-82}$$

Note that unlike the situation wherein the array adapts to the fading, the mean squared error does not vanish when the thermal noise vanishes:

$$\frac{\overline{|e|^2}}{S} \rightarrow 1-2m^2 \frac{(1 - \mathrm{Re}\{\rho_0\}) + \frac{J}{S}(1 - \mathrm{Re}\{\rho_1 e^{j\pi\sin\theta}\})}{(1 - |\rho_0|^2) + (\frac{J}{S})^2(1 - |\rho_1|^2) + 2(\frac{J}{S})(1 - \mathrm{Re}\{\rho_0^* \rho_1 e^{j\pi\sin\theta}\})} \tag{4-83}$$

For large J/S ratios this expression further simplifies to

$$\frac{\overline{|e|^2}}{S} \rightarrow 1-2|m|^2 \quad \frac{S}{J}\left[\frac{1 - \mathrm{Re}\{\rho_1 e^{j\pi\sin\theta}\}}{1 - |\rho_1|^2}\right] \tag{4-84}$$

which shows the mean squared error increasing as J/S increases. This may be contrasted to the case of fading adaptation where the error reaches a minimum at high J/S ratios. The problem here appears due to the term $(J/S)^2(1-|\rho_1|^2)$ which vanishes when spatial decorrelation is absent. It will be shown that the term $(1-|\rho_1|^2)$ is directly proportional to variance of the angle of arrival fluctuations due to the ionospheric scintillation.

SECTION V

SIDELOBE CANCELLER EXAMPLES

In this section we apply the results of Section 3 to the case of a two-element SLC (sidelobe canceller) as shown in Figure 5-1. We assume sufficiently narrowband transmission that frequency selective distortion may be ignored. As in the case of the fully adaptive array of Section 4 we consider three cases in order: non-fading, fading with array adaptation to the fading, and fading with array non-adaptation to fading.

To simplify the analysis we shall confine our attention to the case wherein the signal level received in the sidelobe canceller antennas is small enough relative to the jammer level that it may be neglected.

5.1 NON-FADING CASE

Using the above assumptions and unity gain channels, the signals received in the three antenna beams of Figure 5-1 are expressible in the form

$$w_0(t) = A_0(\theta_0)s(t) + A_0(\theta_1)j_1(t) + A_0(\theta_2)j_2(t) + n_0(t) \tag{5-1}$$

$$w_1(t) = A_1(\theta_1)e^{-j2\pi\frac{d}{\lambda}\sin\theta_1} j_1(t) + A_1(\theta_2)e^{-j2\pi\frac{d}{\lambda}\sin\theta_2} j_2(t) + n_1(t) \tag{5-2}$$

$$w_2(t) = A_2(\theta_1)e^{j2\pi\frac{d}{\lambda}\sin\theta_1} j_1(t) + A_2(\theta_2)e^{j2\pi\frac{d}{\lambda}\sin\theta_2} j_2(t) + n_2(t) \tag{5-3}$$

where $A_k(\theta)$ is the complex antenna pattern for the k^{th} antenna as a function of the angle measured relative to boresight (see Figure 5-1) and θ_0, θ_1, and

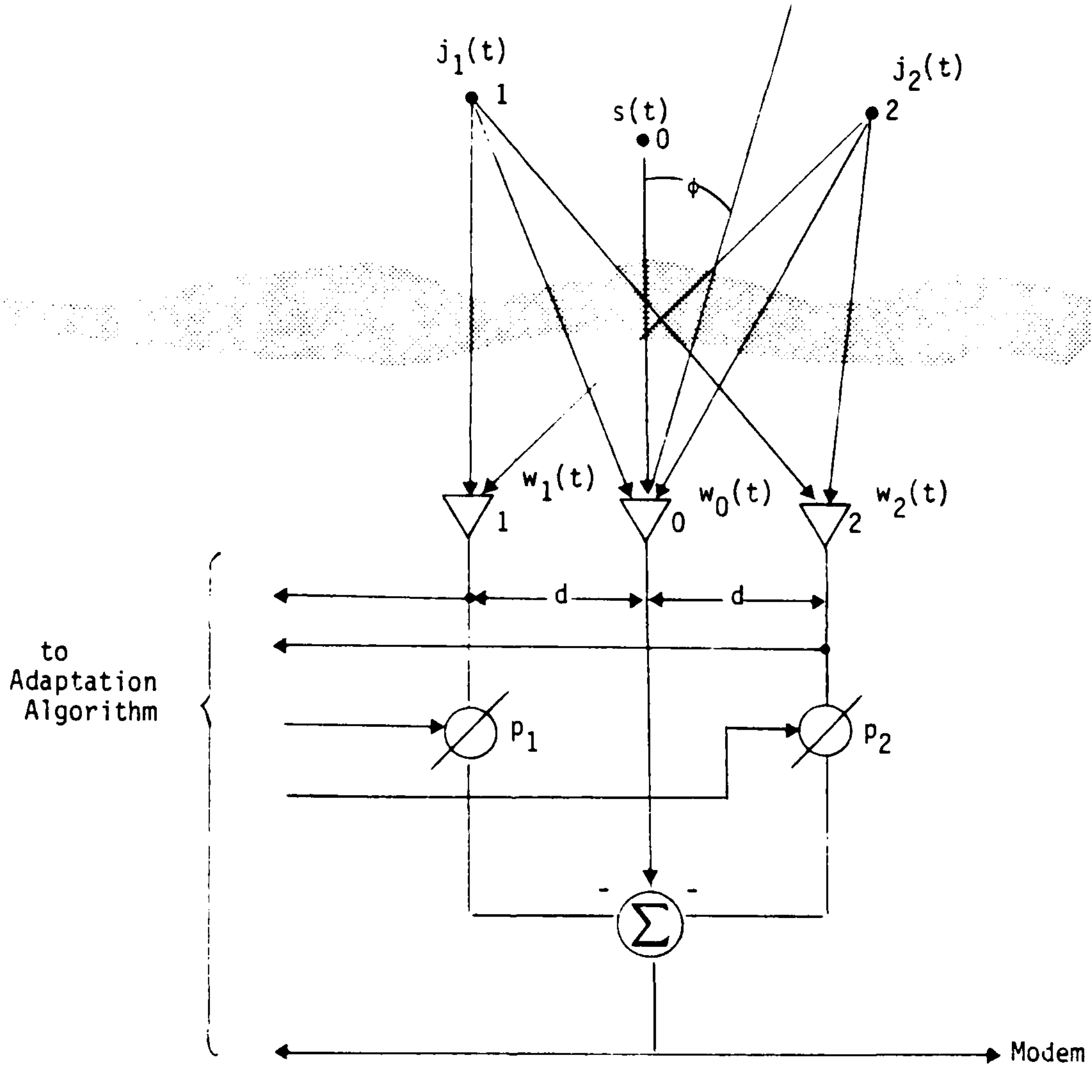

Figure 5-1 Two-Element Example of Sidelobe Canceller

θ_2 are the angular positions of the signal, jammer 1 and jammer 2 transmitters. The distance between the electrical centers of each of the sidelobe antenna canceller beams and the main antenna is d. To simplify notation we shall assume identical canceller beam patterns

$$A_1(\phi) = A_2(\phi) = A(\phi) \tag{5-4}$$

As discussed in Section 2, in the case of the power nulling system the reference signal r(t) is set equal to the main beam output, i.e.,

$$r(t) = w_0(t) \tag{5-5}$$

Thus the crosscorrelation vector $\underline{R}$ [see Eq.(3-3)] is given by

$$\underline{R} = \left[\overline{w_0 w_1^*} \, , \, \overline{w_0 w_2^*} \right] \tag{5-6}$$

Computing the averages we find that

$$\underline{R}^T = \begin{bmatrix} H_{01} e^{j\psi_1/2} J_1 + H_{02} e^{j\psi_2/2} J_2 \\ H_{01} e^{-j\psi_1/2} J_1 + H_{02} e^{-j\psi_2/2} J_2 \end{bmatrix} \tag{5-7}$$

where we have used the definitions

$$H_{0n} = A_0(\theta_n) \, A^*(\theta_n) \tag{5-8}$$

$$\psi_n = 4\pi \frac{d}{\lambda} \sin \theta_n \tag{5-9}$$

and J_n is the strength of $j_n(t)$.

The next quantity needed is the correlation matrix C [see Eq.(3-4)] which we readily find to be

$$C = \begin{bmatrix} G_1J_1 + G_2J_2 + N & G_1J_1e^{j\psi_1} + G_2J_2e^{j\psi_2} \\ G_1J_1e^{-j\psi_1} + G_2J_2e^{-j\psi_2} & G_1J_1 + G_2J_2 + N \end{bmatrix} \tag{5-10}$$

in which we have used the definition

$$G_n = |A_0(\theta_n)|^2 \tag{5-11}$$

The inverse correlation matrix is given by

$$C^{-1} = \frac{1}{\Delta} \begin{bmatrix} G_1J_1 + G_2J_2 + N & -G_1J_1e^{j\psi_1} - G_2J_2e^{j\psi_2} \\ -G_1J_1e^{-j\psi_1} - G_2J_2e^{-j\psi_2} & G_1J_1 + G_2J_2 + N \end{bmatrix} \tag{5-12}$$

where

$$\Delta = 4\, G_1G_2J_1J_2\sin^2\left(\frac{\psi_1 - \psi_2}{2}\right) + 2N(G_1J_1 + G_2J_2) + N^2 \tag{5-13}$$

The optimum weights are obtained from Eq.(3-1) as

$$\underline{P} =$$

$$\frac{1}{\Delta}\begin{bmatrix} G_1J_1 + G_2J_2 + N & -(G_1J_1E^{j\psi_1} + G_2J_2E^{j\psi_2}) \\ -(G_1J_1e^{-j\psi_1} + G_2J_2e^{-j\psi_2}) & G_1J_1 + G_2J_2 + N \end{bmatrix} \begin{bmatrix} H_{01}J_1e^{j\psi_1/2} + H_{02}J_2e^{j\psi_2/2} \\ H_{01}J_1e^{-j\psi_1/2} + H_{02}J_2e^{-j\psi_2/2} \end{bmatrix} \tag{5-14}$$

Carrying out the indicated operations we find

$$\underline{P} =$$

$$\frac{1}{\Delta}\begin{bmatrix} 2J_1J_2 j \sin\left(\frac{\psi_2-\psi_1}{2}\right)\left(G_1H_{02}e^{j\psi_1/2} - G_2H_{01}e^{j\psi_2/2}\right) + N\left(H_{01}J_1e^{j\psi_1/2} + H_{02}J_2e^{j\psi_2/2}\right) \\ 2J_1J_2 j \sin\left(\frac{\psi_2-\psi_1}{2}\right)\left(G_2H_{01}e^{-j\psi_2/2} - G_1H_{02}e^{-j\psi_1/2}\right) + N\left(H_{01}J_1e^{-j\psi_1/2} + H_{02}J_2e^{-j\psi_2/2}\right) \end{bmatrix} \tag{5-15}$$

The minimum mean squared error (minimum output power in the present case) is given by Eq.(3-6). After carrying out the operations indicated we find that

$$\overline{|e|^2} = G_{00}S + N + 2\,N\,D \tag{5-16}$$

in which

$$D = \frac{\left(\frac{J_1}{N}\right)\left(\frac{J_2}{N}\right)\left(G_{10}G_2 + G_{20}G_1 - 2\cos\left(\frac{\psi_1-\psi_2}{2}\right)\mathrm{Re}\{H_{01}^{*}H_{02}\}\right) + \left(G_{10}\left(\frac{J_1}{N}\right) + G_{20}\left(\frac{J_2}{N}\right)\right)}{4\left(\frac{J_1}{N}\right)\left(\frac{J_2}{N}\right)G_1G_2\sin^2\left(\frac{\psi_1-\psi_2}{2}\right) + 2\left(G_1\left(\frac{J_1}{N}\right) + G_2\left(\frac{J_2}{N}\right)\right) + 1} \tag{5-17}$$

and we have used the definitions

$$G_{n0} = |A_0(\theta_n)|^2 \quad : \quad n = 1,2 \tag{5-18}$$

In the case of a single jammer (5-16) simplifies to

$$\overline{|e|^2} = G_{00} S + N + 2N \frac{G_{10}\left(\frac{J_1}{N}\right)}{2G_1\left(\frac{J_1}{N}\right) + 1} \tag{5-19}$$

The first term in (5-16) is the signal output power while the remaining terms constitute the output noise and jamming level. Thus the output SNR is given by

$$\rho = G_{00} \frac{S}{N} \frac{1}{1 + 2D} \tag{5-20}$$

For large J/N ratios ($\psi_1 \neq \psi_2$)

$$D \rightarrow \frac{|A_0(\theta_1)|^2|A(\theta_2)|^2+|A_0(\theta_2)|^2|A(\theta_1)|^2-2\mathrm{Re}\{A_0^*(\theta_1)A(\theta_1)A_0(\theta_2)A^*(\theta_2)\}\cos\left(\frac{\psi_1-\psi_2}{2}\right)}{4|A(\theta)|^2|A(\theta_2)|^2\sin^2\left(\frac{\psi_1-\psi_2}{2}\right)} \tag{5-21}$$

Note that D is directly proportional to the level of jamming power entering the main lobe (i,.e., the gain of the sidelobes of the main beam). If we assume the ideal condition that the complex gain of the canceller antenna is set equal to the complex gain of the sidelobes of the main beams and these gains are independent of jammer position

$$A_0(\theta_1) = A_0(\theta_2) = A(\theta_1) = A(\theta_2) \tag{5-22}$$

Then we see that D = 1 and

$$\rho = \frac{G_{00} S}{3N} \quad : \quad (A_0 = A) \tag{5-23}$$

Note that the SNR has decreased by a factor of 3 over an unjammed case.

On the other hand if the sidelobes of the main beam are much weaker than the gain of the canceller beams, i.e.,

$$|A_0(\theta_n)| \ll A(\theta_m) \quad : \quad n,m = 1,2 \tag{5-24}$$

then we see that

$$\rho \rightarrow G_{00} \frac{S}{N} \qquad (|A_0| \ll A) \tag{5-25}$$

and no loss in output SNR will occur relative to the unjammed case.

In general, at large J/N ratios, the jammers are cancelled but the resulting thermal noise level will vary depending upon the size of the main beam sidelobe gains relative to the canceller beam gains.

5.2 FLAT FADING WITH SPATIAL DECORRELATION: ARRAY ADAPTIVE TO FADING

In this section we assume the channels from the two jammers and the signal to the three elements, nine channels in all, are subjected to flat fading. Moreover we assume that the channels from a given transmitter to the different antennas may fade differently. The received process in the main beam, the reference signal, may be expressed in the general form

$$w_0(t) = h_{00}s(t) + h_{10}j_1(t) + h_{20}j_2(t) + n_0(t) \qquad (5\text{-}26)$$

where, for simplicity, we have incorporated the effect of the main beam antenna pattern on the received signals within the definition of the complex channel gains $h_{00}(t)$, $h_{11}(t)$, and $h_{20}(t)$.

Similarly the received processes in the two sidelobe canceller beams are

$$w_1(t) = h_{11}j_1(t) + h_{21}j_2(t) + n_1(t) \qquad (5\text{-}27)$$

$$w_2(t) = h_{12}j_1(t) + h_{22}j_2(t) + n_2(t) \qquad (5\text{-}28)$$

where the antenna patterns of the canceller beams and any phase shifts caused by antenna spacing are incorporated in the $\{h_{mn}(t) : m,n = 1,2\}$.

Relating the notation in the present and previous sections,

$$h_{00}(t) = A_0(\theta_0)\, g_{00}(t) \tag{5-29}$$

$$h_{10}(t) = A_0(\theta_1)\, g_{10}(t) \tag{5-30}$$

$$h_{20}(t) = A_0(\theta_2)\, g_{20}(t) \tag{5-31}$$

$$h_{11}(t) = A_1(\theta_1)\, e^{-j\psi_1/2}\, g_{11}(t) \tag{5-32}$$

$$h_{21}(t) = A_1(\theta_2)\, e^{-j\psi_2/2}\, g_{21}(t) \tag{5-33}$$

$$h_{12}(t) = A_2(\theta_1)\, e^{j\psi_1/2}\, g_{12}(t) \tag{5-34}$$

$$h_{22}(t) = A_2(\theta_2)\, e^{j\psi_2/2}\, g_{22}(t) \tag{5-35}$$

where $g_{mn}(t)$ represent the complex channel gains that have been normalized (without loss of generality) to unit average squared magnitude

$$\overline{|g_{mn}|^2} = 1 \tag{5-36}$$

Using (3-2) and (3-3) with $r(t) = w_0(t)$ and carrying out the indicated averages

$$R = \begin{bmatrix} h_{10}h_{11}^*J_1 + h_{20}h_{21}^*J_2 \\ h_{10}h_{12}^*J_1 + h_{20}h_{22}^*J_2 \end{bmatrix} \tag{5-37}$$

$$C = \begin{bmatrix} |h_{11}|^2J_1 + |h_{21}|^2J_2 + N & h_{11}^*h_{12}J_1 + h_{21}^*h_{22}J_2 \\ h_{11}h_{12}^*J_1 + h_{21}h_{22}^*J_2 & |h_{12}|^2J_1 + |h_{22}|^2J_2 + N \end{bmatrix} \tag{5-38}$$

where we have assumed that the two jammers are statistically independent and of strengths J_1, J_2.

The inverse to C is given by

$$C^{-1} = \frac{1}{\Delta} \begin{bmatrix} |h_{12}|^2J_1 + |h_{22}|^2J_2 + N & -(h_{11}^*h_{12}J_1 + h_{21}^*h_{22}J_2) \\ -(h_{11}h_{12}^*J_1 + h_{21}h_{22}^*J_2) & |h_{11}|^2J_1 + |h_{21}|^2J_2 + N \end{bmatrix} \tag{5-39}$$

where

$$\Delta = |h_{11}h_{22}-h_{12}h_{21}|^2 J_1J_2 + (|h_{11}|^2+|h_{12}|^2)NJ_1 + (|h_{21}|^2+|h_{22}|^2)NJ_2 + N^2 \tag{5-40}$$

The optimum weights for jammer cancellation are then [from 4-70)]

$$P = \frac{1}{\Delta} \begin{bmatrix} |h_{12}|^2J_1 + |h_{22}|^2J_2 + N & -(h_{11}^*h_{12}J_1 + h_{21}^*h_{12}J_2) \\ -(h_{11}h_{12}^*J_1 + h_{21}h_{22}^*J_2) & |h_{11}|^2J_1 + |h_{21}|^2J_2 + N \end{bmatrix} \begin{bmatrix} h_{10}h_{11}^*J_1 + h_{20}h_{21}^*J_2 \\ h_{10}h_{12}^*J_1 + h_{20}h_{22}^*J_2 \end{bmatrix} \tag{5-41}$$

Carrying out the algebra

$$P = \frac{1}{\Delta}\begin{bmatrix} (h_{22}^{*}h_{11}^{*} - h_{21}^{*}h_{12}^{*})(h_{22}h_{10}-h_{12}h_{20})J_1J_2 + h_{10}h_{11}^{*}J_1N + h_{20}h_{21}^{*}J_2N \\ (h_{22}^{*}h_{11}^{*} - h_{21}^{*}h_{12}^{*})(h_{11}h_{20}-h_{21}h_{10})J_1J_2 + h_{10}h_{12}^{*}J_1N + h_{20}h_{22}^{*}J_2N \end{bmatrix} \tag{5-42}$$

Continuing on to compute the minimum mean squared error (in the present context this would be the minimum array output power) using Eq.(3-6) we find after considerable algebra that

$$\overline{|e|^2} = |h_{00}|^2 S + N + N\,\frac{\left(\frac{J_1}{N}\right)\left(\frac{J_2}{N}\right)(\Delta_{10}+\Delta_{02}) + \left(|h_{10}|^2\left(\frac{J_1}{N}\right) + |h_{20}|^2\left(\frac{J_2}{N}\right)\right)}{\left(\frac{J_1}{N}\right)\left(\frac{J_2}{N}\right)\Delta_{12} + \left(\frac{J_1}{N}\right)(|h_{11}|^2+|h_{12}|^2) + \left(\frac{J_2}{N}\right)(|h_{21}|^2+|h_{22}|^2)+1} \tag{5-43}$$

where

$$\Delta_{12} = |h_{11}h_{22} - h_{12}h_{21}|^2 \tag{5-44}$$

$$\Delta_{01} = |h_{10}h_{21} - h_{20}h_{11}|^2 \tag{5-45}$$

$$\Delta_{02} = |h_{10}h_{22} - h_{20}h_{12}|^2 \tag{5-46}$$

The first term in (5-43) represents the output signal power while the second two terms represent the output noise and jamming power. It follows that the output SNR ρ is given by

$$\rho = |h_{00}|^2\,\frac{S}{N}\,\frac{\left(\frac{J_1}{N}\right)\left(\frac{J_2}{N}\right)\Delta_{12} + \left(\frac{J_1}{N}\right)(|h_{11}|^2+|h_{12}|^2) + \left(\frac{J_2}{N}\right)(|h_{21}|^2+|h_{22}|^2)+1}{\left(\frac{J_1}{N}\right)\left(\frac{J_2}{N}\right)(\Delta_{01}+\Delta_{02}+\Delta_{12}) + \frac{J_1}{N}\sum_{n=0}^{2}|h_{1n}|^2 + \frac{J_2}{N}\sum_{n=0}^{2}|h_{2n}|^2 + 1} \tag{5-47}$$

which becomes for large J/N ratios ($\Delta_{12} \neq 0$)

$$\rho \rightarrow \overline{|h_{00}|^2} \; \frac{S}{N} \; \frac{\Delta_{12}}{\Delta_{01} + \Delta_{02} + \Delta_{12}} \quad : \quad \frac{J_1}{N} >> 1 \; , \; \frac{J_2}{N} >> 1 \tag{5-48}$$

The quantities Δ_{mn} are functions of time due to their dependence on the channel fluctuations, so that the resultant SNR (5-48) is also time variant. An interpretation of (5-48) is that at large J/N ratios the jammers are cancelled at the output but the resultant thermal noise is time variable and of level

$$N_{out} = N \left(1 + \frac{\Delta_{01} + \Delta_{02}}{\Delta_{12}} \right) \tag{5-49}$$

For a single jammer we see that

$$\rho = |h_{00}|^2 \frac{S}{N} \; \frac{\frac{J_1}{N} \left(|h_{11}|^2 + |h_{12}|^2 \right) + 1}{\frac{J_1}{N} \left(|h_{10}|^2 + |h_{11}|^2 + |h_{12}|^2 \right) + 1} \tag{5-50}$$

which at large J_1/N values becomes

$$\rho = |h_{00}|^2 \frac{S}{N} \left(\frac{1}{1 + \frac{|h_{10}|^2}{|h_{11}|^2 + |h_{12}|^2)}} \right) \tag{5-51}$$

5.3 FLAT FADING WITH SPATIAL-DECORRELATION: ARRAY NON-ADAPTIVE TO FADING

In this section we will assume flat fading and spatial decorrelation as in Section 5.2 but we now assume that the adaptation circuits cannot follow the fading. Section 5.3.1 considers the case of two jammers and two SLC antennas while Section 5.3.2 examines one jammer with M SLC antennas.

5.3.1 Two Jammers and Two SLC Antennas

The reference crosscorrelation vector R and correlation matrix C now become

$$\underline{R} = \begin{bmatrix} \overline{h_{10}h_{11}^*}J_1 + \overline{h_{20}h_{21}^*}J_2 \\ \overline{h_{10}h_{12}^*}J_1 + \overline{h_{20}h_{22}^*}J_2 \end{bmatrix} \tag{5-52}$$

$$\underline{C} = \begin{bmatrix} \overline{|h_{11}|^2}J_1 + \overline{|h_{21}|^2}J_2 + N & \overline{h_{11}^*h_{12}}J_1 + \overline{h_{21}^*h_{22}}J_2 \\ \overline{h_{11}h_{12}^*}J_1 + \overline{h_{21}h_{22}^*}J_2 & \overline{|h_{12}|^2}J_1 + \overline{|h_{22}|^2}J_2 + N \end{bmatrix} \tag{5-53}$$

Using (5-29) through (5-36), the averages in (5-52) and (5-53) may be related to jammer positions, antenna patterns, and fading correlation coefficients as follows

$$\overline{h_{10}h_{11}^*} = A_0(\theta_1)\, A_1^*(\theta_1)\, e^{j\psi_1/2}\, \rho_{11,10} \tag{5-54}$$

$$\overline{h_{20}h_{21}^*} = A_0(\theta_2)\, A_1^*(\theta_2)\, e^{j\psi_2/2}\, \rho_{21,20} \tag{5-55}$$

$$\overline{h_{10}h_{12}^*} = A_0(\theta_1)\, A_2^*(\theta_1)\, e^{-j\psi_1/2}\, \rho_{12,10} \tag{5-56}$$

$$\overline{h_{20}h_{22}^*} = A_0(\theta_2)\, A_2^*(\theta_2)\, e^{-j\psi_2/2}\, \rho_{22,20} \tag{5-57}$$

$$\overline{|h_{11}|^2} = |A_1(\theta_1)|^2 \tag{5-58}$$

$$\overline{|h_{21}|^2} = |A_1(\theta_2)|^2 \tag{5-59}$$

$$\overline{|h_{12}|^2} = |A_2(\theta_1)|^2 \tag{5-60}$$

$$\overline{|h_{22}|^2} = |A_2(\theta_2)|^2 \tag{5-61}$$

$$\overline{h_{11}^* h_{12}} = A_1^*(\theta_1)\ A_2(\theta_1)\ e^{j\psi_1}\ \rho_{11,12} \tag{5-62}$$

$$\overline{h_{21}^* h_{22}} = A_1^*(\theta_2)\ A_2(\theta_2)\ e^{j\psi_2}\ \rho_{21,22} \tag{5-63}$$

where we have defined the correlation coefficient

$$\rho_{mn,pq} = \frac{\overline{g_{mn}^* g_{pq}}}{\sqrt{\overline{|g_{mn}|^2}\ \ \overline{|g_{pq}|^2}}} \tag{5-64}$$

The correlation matrix inverse is given by

$$\underline{C}^{-1} = \frac{1}{\Delta} \begin{bmatrix} \overline{|h_{12}|^2} J_1 + \overline{|h_{22}|^2} J_2 + N & -(\overline{h_{11}^* h_{12}} J_1 + \overline{h_{21}^* h_{22}} J_2) \\ -(\overline{h_{11} h_{12}^*} J_1 + \overline{h_{21} h_{22}^*} J_2\ ) & \overline{|h_{11}|^2} J_1 + \overline{|h_{21}|^2} J_2 + N \end{bmatrix} \tag{5-65}$$

where Δ is the determinant of C

$$\Delta = (\overline{|h_{12}|^2}\ \overline{|h_{11}|^2} - |\overline{h_{11}^* h_{12}}|^2)J_1^2 + (\overline{|h_{22}|^2}\ \overline{|h_{21}|^2} - |\overline{h_{21}^* h_{22}}|^2)\ J_2^2$$

$$+ (\overline{|h_{22}|^2}\ \overline{|h_{22}|^2} + \overline{|h_{12}|^2}\ \overline{|h_{21}|^2} - \overline{h_{21}^* h_{22}} \cdot \overline{h_{11} h_{12}^*} - \overline{h_{11}^* h_{12}} \cdot \overline{h_{21} h_{22}^*})J_1 J_2$$

$$+ N(\overline{|h_{11}|^2} + \overline{|h_{12}|^2})J_1 + N(\overline{|h_{22}|^2} + \overline{|h_{21}|^2}) + N^2 \tag{5-66}$$

Using Eqs.(5-54) through (5-63) to relate the averages in (5-66) to the antenna gains and channel correlation coefficients

$$\Delta = G_{12}G_{11}(1 - |\rho_{11,12}|^2)J_1^2 + G_{21}G_{22}(1 - |\rho_{21,22}|^2)\ J_2^2$$

$$+ \left[G_{22}G_{11} + G_{21}G_{12} - 2\mathrm{Re}\{A_{21}^* A_{22} A_{11} A_{12}^* \rho_{21,22} \rho_{12,11}^*\}\right] J_1 J_2$$

$$+ N(G_{11}+G_{12})J_1 + N(G_{21}+G_{22})J_2 + N^2 \tag{5-67}$$

where

$$G_{mn} = |A_{mn}|^2 \tag{5-68}$$

$$A_{mn} = A_n(\theta_m) \tag{5-69}$$

Expressions for the optimum weights, mean squared error, and SNR may be obtained as in the previous sections but these do not simplify as in the previous case and we do not present them. A subsequent memo will present results based on computer evaluation. However, to obtain some understanding of the effect of spatial decorrelation without using a computer, it is sufficient to consider special cases. Consider first the case of a single jammer. Then (5-52), (5-53) and (5-65) simplify to

$$\underline{R} = \begin{bmatrix} \overline{h_{10}h_{11}^*}\, J \\ \overline{h_{10}h_{12}^*}\, J \end{bmatrix} \tag{5-70}$$

$$\underline{C} = \begin{bmatrix} \overline{|h_{11}|^2}\, J + N & \overline{h_{11}^*h_{12}}\, J \\ \overline{h_{11}h_{12}^*}\, J & \overline{|h_{12}|^2}\, J \end{bmatrix} \tag{5-71}$$

$$C^{-1} = \frac{1}{\Delta} \begin{bmatrix} \overline{|h_{11}|^2}\, J + N & -\overline{h_{11}^*h_{12}}\, J \\ -\overline{h_{11}h_{12}^*}\, J & \overline{|h_{11}|^2}\, J + N \end{bmatrix} \tag{5-72}$$

where

$$\Delta = J^2 \left(\overline{|h_{11}|^2}\;\overline{|h_{12}|^2} - \overline{|h_{11}^*h_{12}|^2}\right) + N\, J\left(\overline{|h_{12}|^2} + \overline{|h_{11}|^2}\right) + N \tag{5-73}$$

Using (3-1) and (3-4), the optimum weights, minimum output power and output SNR are

$$\underline{p} = \frac{1}{\Delta} \begin{bmatrix} (\overline{|h_{12}|^2}\,\overline{h_{10}h_{11}^*} - \overline{h_{11}^* h_{12}}\,\overline{h_{10}h_{12}^*})\, J + N\, \overline{h_{10}h_{11}^*} \\ (\overline{|h_{11}|^2}\,\overline{h_{10}h_{12}^*} - \overline{h_{11}h_{12}^*}\,\overline{h_{10}h_{12}^*})\, J + N\, \overline{h_{10}h_{12}^*} \end{bmatrix} \tag{5-74}$$

$$\overline{|e|^2} = \overline{|h_{00}|^2} S + N + N \frac{\left(\frac{J}{N}\right)^3 D_1 + \left(\frac{J}{N}\right)^2 D_2 + \left(\frac{J}{N}\right) \overline{|h_{11}|^2}}{\left(\frac{J}{N}\right)^2 D_3 + \left(\frac{J}{N}\right)\left(\overline{|h_{12}|^2} + \overline{|h_{11}|^2}\right) + 1} \tag{5-75}$$

$$\rho = \overline{|h_{00}|^2}\ \frac{S}{N} \frac{\left(\frac{J}{N}\right)^2 D_3 + \left(\frac{J}{N}\right)\left(\overline{|h_{12}|^2} + \overline{|h_{11}|^2}\right) + 1}{\left(\frac{J}{N}\right)^3 D_1 + \left(\frac{J}{N}\right)^2 D_4 + \left(\frac{J}{N}\right) D_5 + 1} \tag{5-76}$$

where

$$D_1 = \overline{|h_{10}|^2}\,\overline{|h_{11}|^2}\,\overline{|h_{12}|^2} - \overline{|h_{10}|^2}\left|\overline{h_{11}^* h_{12}}\right|^2 - \overline{|h_{12}|^2}\left|\overline{h_{10}^* h_{11}}\right|^2$$
$$- \overline{|h_{11}|^2}\left|\overline{h_{10}^* h_{12}}\right|^2 + 2\,\mathrm{Re}\{\overline{h_{10}^* h_{11}} \cdot \overline{h_{11}^* h_{12}} \cdot \overline{h_{10} h_{12}^*}\} \tag{5-77}$$

$$D_2 = \overline{|h_{10}|^2}\,\overline{|h_{12}|^2} - \left|\overline{h_{10}h_{12}^*}\right|^2 + \overline{|h_{10}|^2}\,\overline{|h_{11}|^2} - \left|\overline{h_{10}h_{11}^*}\right|^2 \tag{5-78}$$

$$D_3 = \overline{|h_{11}|^2}\;\overline{|h_{12}|^2} - \left|\overline{h_{12}h_{12}^*}\right|^2 \tag{5-79}$$

$$D_4 = D_2 + D_3 \tag{5-80}$$

$$D_5 = \overline{|h_{10}|^2} + \overline{|h_{12}|^2} + \overline{|h_{11}|^2} \tag{5-81}$$

and

$$\overline{|h_{10}|^2} = |A_0(\theta_1)|^2 \tag{5-82}$$

Examining the SNR (5-63) for large J/S ratios we see that

$$\rho \rightarrow \overline{|h_{00}|^2}\;\frac{S}{J}\cdot\frac{D_3}{D_1} \tag{5-83}$$

Thus, as in the fully adaptive array (Section 4.3), when the adaptive array does not adapt to the fading, the jammer can be reduced but not cancelled and the output SNR varies as the input S/J ratio for large J/N ratios.

The "processing gain" against the jammer (ratio of output to input S/J ratio) is just

$$g = \frac{D_3}{\overline{|g_{00}|^2}\;D_1} \tag{5-84}$$

since the input signal/jamming power ratio is

$$(S/J)_{input} = \frac{\overline{|g_{00}|^2}}{\overline{|g_{10}|^2}}\;\frac{S}{J} \tag{5-85}$$

The presence of spatial decorrelation causes the term D_1 to be non-vanishing and results in the poor asymptotic behavior (5-71). To see this, we may use Eqs. (5-54) through (5-63) and (5-70) to express the coefficients D_1 in terms of antenna gains and channel correlation coefficients. The result is

$$D_1 = G_{10}G_{11}G_{12}\left[1 - |\rho_{11,12}|^2 - |\rho_{10,11}|^2 - |\rho_{10,12}|^2 + 2\,\mathrm{Re}\{\rho^*_{11,10}\,\rho_{11,12}\,\rho^*_{10,12}\}\right] \tag{5-86}$$

Expressions follow for D_2 and D_3 (and thus D_4) which also exhibit the vanishing behavior with the absence of spatial decorrelation.

$$D_2 = G_{10}G_{12}\left[1 - |\rho_{12,10}|^2\right] + G_{11}G_{10}\left[1 - |\rho_{11,10}|^2\right] \tag{5-87}$$

$$D_3 = G_{11}G_{12}\left[1 - |\rho_{11,12}|^2\right] \tag{5-88}$$

Thus the processing gain may be expressed as

$$g = \frac{1 - |\rho_{11,12}|^2}{1 - |\rho_{11,12}|^2 - |\rho_{10,11}|^2 - |\rho_{10,12}|^2 + 2\,\mathrm{Re}\{\rho^*_{11,10}\,\rho_{11,12}\,\rho^*_{10,12}\}} \tag{5-89}$$

An even simpler case worth examining is the case of a single canceller antenna and a single jammer. In this case we assume $g_{12} = 0$, $J_2 = 0$. Then in Eq.(5-75), $D_3 = D_1 = 0$ and

$$\rho = G_{00} \frac{S}{N} \frac{\left(\frac{J}{N}\right) G_{11} + 1}{\left(\frac{J}{N}\right)^2 G_{10}G_{11}(1 - |\rho_{10,11}|^2) + \left(\frac{J}{N}\right)(G_{10}+G_{11}) + 1} \tag{5-90}$$

Again we observe the same type of behavior at large J/N ratios

$$\rho \to \frac{G_{00}}{G_{10}} \frac{S}{J} \frac{1}{1 - |\rho_{10,11}|^2} \tag{5-91}$$

Thus the processing gain at high jamming levels is

$$g = \frac{1}{1 - |\rho_{10,11}|^2} \tag{5-92}$$

5.3.2 One Jammer and M SLC Antennas

We now derive a general expression involving only channel correlation coefficients for the high J/S ratio processing gain against one jammer when M s delobe cancelling antennas are used. With a single jammer

$$w_0(t) = h_{00}(t)\, s(t) + h_{10}(t)\, j(t) + n_0(t) \tag{5-93}$$

$$w_m(t) = h_{1m}(t)\, j(t) + n_m(t) \quad ; \quad m = 1,2,\ldots M \tag{5-94}$$

The reference signal crosscorrelation vector and the input process covariance matrix are given by

$$\underline{R} = J \begin{bmatrix} \overline{h_{10}h_{11}^*} \\ \overline{h_{10}h_{12}^*} \\ \vdots \\ \overline{h_{10}h_{1M}^*} \end{bmatrix} \tag{5-95}$$

$$\underline{C} = \begin{bmatrix} \overline{|h_{11}|^2}\, J + N & \overline{h_{11}^* h_{12}}\, J & \cdots & \overline{h_{11}^* h_{1M}} J \\ \overline{h_{12}^* h_{11}}\, J & \overline{|h_{12}|^2} J + N & \cdots & \\ \vdots & & & \\ \overline{h_{1M}^* h_{11}}\, J & & \cdots & \overline{|h_{1M}|^2}\, J+N \end{bmatrix} \tag{5-96}$$

In terms of antenna gain patterns and channel correlation coefficients

$$\overline{h_{10}h^*_{1m}} = A_0(\theta_1)\, A^*_m(\theta_1)\, e^{j\gamma_m}\, \rho_{m0} \tag{5-97}$$

$$\overline{h^*_{1m}h_{1p}} = A_p(\theta_1)\, A^*_m(\theta_1)\, e^{-j(\gamma_p-\gamma_m)}\, \rho_{mp} \tag{5-98}$$

where we have used the simpler notation

$$\rho_{1m,10} = \rho_{m0} \tag{5-98}$$

$$\rho_{1m,1p} = \rho_{mp} \tag{5-100}$$

and defined γ_m as the phase shift in the received jamming signal at the m^{th} antenna relative to that at the main antenna due to differential path lengths.

Using (5-98) in (5-96) and assuming that the jamming power is large enough relative to the noise that N may be neglected in (5-96), we see that the typical term in (5-86) may be expressed in the form

$$\underline{C} = \{A_p(\theta_1)\, A^*_m(\theta_1)\, e^{-j(\gamma_p-\gamma_m)}\, \rho_{mp}\} \tag{5-101}$$

where

$$\rho_{mm} = 1 \qquad ; \quad m = 1,2,\ldots,M \tag{5-102}$$

As a consequence of (5-101) we may factor $\underline{C}$ into the form

$$\underline{C} = J\, \underline{A}^H \underline{r}_M \underline{A} \qquad (N=0) \tag{5-103}$$

where $\underline{A}^H$ is a diagonal matrix

$$\underline{A}^H = \begin{bmatrix} A_1^*(\theta_1)e^{j\gamma_1} & 0 & \cdots & 0 \\ 0 & A_2^*(\theta_1)e^{j\gamma_2} & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & A_M^*(\theta_1)e^{j\gamma_M} \end{bmatrix} \tag{5-104}$$

and $\underline{r}$ is a matrix of crosscorrelation coefficients

$$\underline{r}_M = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1,M} \\ \rho_{12}^* & 1 & \cdots & \\ \vdots & & & \\ \rho_{1M}^* & \cdots & & 1 \end{bmatrix} \tag{5-105}$$

Using (5-97), (5-95) becomes

$$\underline{R} = J\, A_0(\theta_1) \begin{bmatrix} A_1^*(\theta_1)e^{j\gamma_1}\ \rho_{10} \\ A_2^*(\theta_1)e^{j\gamma_2}\ \rho_{20} \\ \vdots \\ A_M^*(\theta_1)e^{j\gamma_M}\ \rho_{M0} \end{bmatrix} \tag{5-106}$$

The minimum output power is [Eq.(3-5) with the reference signal $r(t) = w_0(t)$]

$$\overline{|e|^2} = \overline{|h_{00}|^2}\, S + \overline{|h_{10}|^2}\, J + N - R^H \; C^{-1} \; R \quad c \tag{5-107}$$

and the output SNR

$$\rho = \frac{\overline{|h_{00}|^2}\; S}{\overline{|h_{10}|^2}\, J + N - R^H \; C^{-1} \; R} \tag{5-108}$$

Using (5-85) and letting $N \to 0$, the expression for the processing gain at high J/S ratios becomes

$$g_M = \frac{\rho}{(S/J)_{input}} = \frac{1}{1 - \frac{1}{J^2} \underline{R}^H \underline{A}^{-1} \underline{r}^{-1} (\underline{A}^H)^{-1} \underline{R}} \tag{5-109}$$

Carrying out the operations indicated in (5-109) we find

$$g_M = \frac{1}{1 - \underline{\rho}_M^H \underline{r}_M^{-1} \underline{\rho}_M} \tag{5-110}$$

where

$$\rho_M^H = \left[\rho_{10}^*, \rho_{20}^*, \dots \rho_{M0}^* \right] \tag{5-111}$$

SECTION VI

SOME AJ PROCESSING GAIN RESULTS AGAINST A SINGLE JAMMER FOR THE FADING NON-ADAPTIVE SIDELOBE CANCELLER

6.1 INTRODUCTION

In order to obtain numerical results on AJ performance using the theory developed in Section 5.3.2, it is necessary to specify an SLC antenna array configuration and a spatial correlation function.

For an arbitrary direction of arrival and an arbitrary antenna configuration, the electrical centers of the SLC antennas will not all fall in a plane. However, the correlation distance along the direction of propagation is very much larger than that in the perpendicular plane. We shall assume that for the size antennas of interest the dimensions along the propagation direction will be small enough to produce negligible spatial decorrelation. As a result, for purposes of calculating the spatial correlation coefficients, it is sufficient to project the electrical centers of the antennas into a plane perpendicular to the direction of propagation. The distance between electrical centers in the <u>projected</u> antenna are used in the computation of correlation coefficients.

As an example, consider the linear array shown in Figure 6-1. The angle of arrival of the jammer (in the absence of scintillation) is θ relative to boresight. Antenna numbers 1,2,...,M are the SLC antennas spaced d units apart while the mainbeam antenna is located at position 0. The projected antenna is shown in dashed lines. For the projected antenna the elements are spaced $d \cos \theta$ units apart.

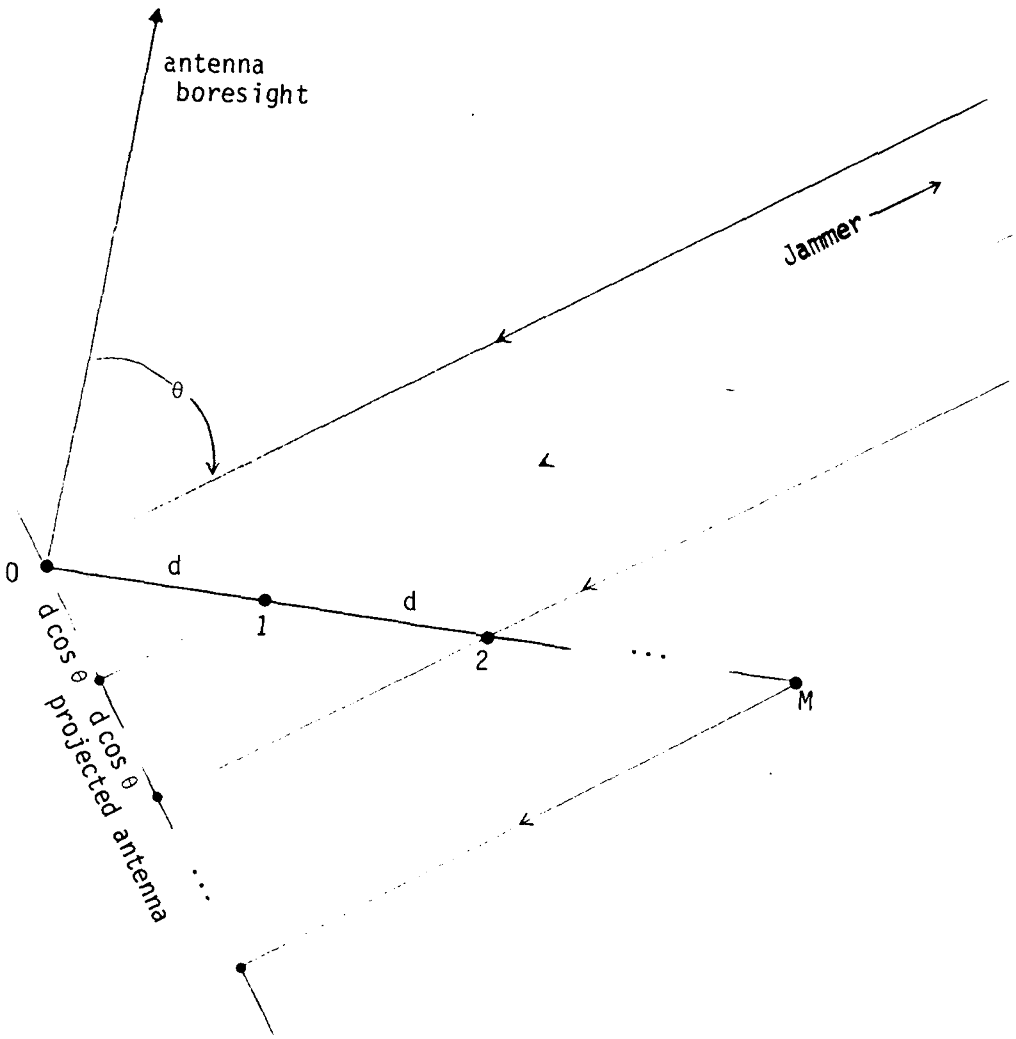

Figure 6-1 Linear Antenna Array Example

In the discussion below we shall initially assume a general spatial correlation function and then one of specific form appropriate to ionospheric scintillations. Section 6.2 derives expressions for the AJ processing gain of an SLC for special cases of interest where the general expressions simplify to a greater or lesser extent. Section 6.3 presents some numerical results.

6.2 DERIVATION OF AJ PROCESSING GAIN FOR SPECIAL CASES

According to Eq.(5-108), the output SNR for a single jammer is given by

$$\rho = \frac{\overline{|h_{00}|^2}\; S}{\overline{|h_{10}|^2}\, J + N - \underline{R}^H\, \underline{C}^{-1}\, \underline{R}} \tag{6-1}$$

where S is the transmitted signal power, J is the transmitted jammer power and N is the additive noise power. The vector $\underline{R}$ is the crosscorrelation between the main beam signal and each of the SLC antenna output signals, and $\underline{C}$ is the crosscorrelation matrix of the SLC antenna output signals. In the absence of the SLC, the output SNR would be

$$\rho_0 = \frac{\overline{|h_{00}|^2}\; S}{\overline{|h_{10}|^2}\, J + N} \tag{6-2}$$

Defining the AJ processing gain as the ratio of output SNR with the adaptive array to output SNR without the array, we find

$$g_M = \frac{\rho}{\rho_0} = \frac{1}{1 - \dfrac{\underline{R}^H\, \underline{C}^{-1}\, \underline{R}}{\overline{|h_{10}|^2}\, J + N}} \tag{6-3}$$

Expressions for the typical terms in $\underline{R}$ and $\underline{C}$ are [see (5-93) through (5-11) modified so that $N \neq 0$],

$$\underline{R} = J\, A_0(\theta)\, \{A_m^*(\theta)\, e^{j\gamma_m}\, \rho_{m0}\;;\; m = 1,2,\ldots M\} \tag{6-4}$$

$$\underline{C} = \{A_p(\theta)\, A_m^*(\theta)\, e^{-j(\gamma_p - \gamma_m)}\, J\rho_{mp} + \delta_{mp}\, N\;;\; m,p = 1,2,\ldots M\} \tag{6-5}$$

where $A_m(\cdot)$ is the antenna pattern for the m^{th} antenna, θ is the direction of arrival of the jammer relative to boresight, γ is the phase delay in the p^{th} antenna relative to the main beam antenna for the jamming signal, ρ_{mp} is the crosscorrelation coefficient between fluctuations of the channels from the jammer to the m^{th} and p^{th} antennas ($\rho_{mm}=1$), and

$$\delta_{mp} = \begin{cases} 1 & ; \quad m = p \\ 0 & ; \quad m \neq p \end{cases} \tag{6-6}$$

As in the case where the noise N was assumed zero [Eq.(5-103)], it is possible to factor $\underline{C}$ as follows

$$\underline{C} = J\,\underline{A}^H\,\underline{r}_M\,\underline{A} \tag{6-7}$$

where A is the diagonal matrix [Eq.(5-104)]

$$\underline{A} = \{A_m(\theta)\,e^{-j\gamma_m}\,\delta_{mp}\ ;\ m,p = 1,2,\ldots M\} \tag{6-8}$$

and $\underline{r}_M$ is now

$$\underline{r}_M = \begin{bmatrix} 1 + \frac{N}{JG_{11}} & \rho_{12} & \cdots & \rho_{1M} \\ \rho_{12}^* & 1 + \frac{N}{JG_{22}} & & \rho_{1,M-1} \\ \vdots & & & \vdots \\ \rho_{1M}^* & \cdots & & 1 + \frac{N}{JG_{MM}} \end{bmatrix} \tag{6-9}$$

where

$$G_{mm} = |A_m(\theta)|^2 \tag{6-10}$$

is the gain pattern of the m^{th} antenna evaluated at angle θ.

Using (6-4) and (6-7) we find that

$$\underline{R}^H \underline{C}^{-1} \underline{R} = \frac{1}{J} \underline{R}^H \underline{A}^{-1} \underline{r}_M^{-1} (A^H)^{-1} \underline{R} \tag{6-11}$$

but

$$\underline{R}^H \underline{A}^{-1} = J A_0^*(\rho) \underline{\rho}_M^H \tag{6-12}$$

where

$$\underline{\rho}_M^H = \left[\rho^*_{10}, \rho^*_{20}, \cdots \underline{\rho}^*_{M0} \right]$$

so that

$$\underline{R}^H \underline{C}^{-1} \underline{R} = J \underline{\rho}_M^H \underline{r}_M^{-1} \underline{\rho}_M |A_0(\theta)|^2 \tag{6-13}$$

Using (6-13) in (6-3) and noting that

$$\overline{|h_{10}|^2} = |A_0(\theta)|^2 \equiv G_0 \tag{6-14}$$

we obtain the general expression for AJ processing gain with one jammer,

$$g_M = \frac{1}{1 - \dfrac{1}{1 + N/JG_0} \underline{\rho}_M^H \underline{r}_M^{-1} \underline{\rho}_M} \tag{6-15}$$

We now consider the simpler but practically interesting case in which the gains of the SLC antennas are all equal. Then we can factor r_M^{-1} and ρ_M in the form

$$r_M = (1 + N/JG_1)\ \hat{\underline{r}}_M \tag{6-16}$$

$$\underline{\rho}_M = (1 + N/JG_1)\ \hat{\underline{\rho}}_M^{-1} \tag{6-17}$$

where

$$\hat{\underline{r}}_M = \{\hat{\underline{\rho}}_{mp}\ ;\ m,p = 1,2,\ldots M\} \tag{6-18}$$

$$\hat{\underline{\rho}}_M = \{\hat{\underline{\rho}}_{0p}\ ;\ m = 1,2,\ldots M\} \tag{6-19}$$

in which

$$\hat{\rho}_{mp} = \begin{cases} \rho_{mp}/(1 + N/JG_1) & ;\quad m \neq p \\ 1 & ;\quad m = p \end{cases} \tag{6-20}$$

and

$$G_1 = G_{mm} \qquad ;\quad m = 1,2,\ldots M \tag{6-21}$$

is the common SLC antenna gain.

Note that (6-18) is the form that r_M takes when the receiver noise is set equal to zero.

Using (6-10), (6-18) and (6-19) in (6-15), the AJ processing gain becomes

$$g_M = \cfrac{1}{1 - \cfrac{\hat{\underline{\rho}}_M^H \, \hat{\underline{r}}_M^{-1} \, \hat{\underline{\rho}}_M}{(1 + N/JG_1)(1 + N/JG_0)}} \tag{6-22}$$

We now consider the further simplifications that arise when the correlation coefficient ρ_{mp} takes the form

$$\rho_{mp} = R_{m-p} = R^*_{p-m} \tag{6-23}$$

This form arises in the case of a uniformly spaced linear array of antennas as shown in Figure 6-1. For this case, $\hat{\underline{r}}_M$ and $\hat{\underline{\rho}}_M$ take the forms

$$\hat{\underline{r}}_M = \begin{bmatrix} 1 & \hat{R}_1 & \hat{R}_2 & \cdots & \hat{R}_{M-1} \\ \hat{R}_1^* & 1 & & & \vdots \\ \vdots & & & & \\ \hat{R}_{M-1}^* & \cdots & & & 1 \end{bmatrix} \tag{6-24}$$

$$\hat{\underline{\rho}}_M^H = \left[\hat{R}_1^*, \hat{R}_2^*, \ldots \hat{R}_M^* \right] \tag{6-25}$$

where

$$\hat{R}_m = R_m / (1 + N/JG_1)$$

In the appendix it is shown that when the forms (6-24) (6-25) are valid

$$\hat{\underline{\rho}}_M^H \, \hat{\underline{r}}_M^{-1} \, \hat{\underline{\rho}}_M = 1 - \frac{\det \hat{r}_{M+1}}{\det \hat{r}_M} \tag{6-26}$$

so that, using (6-26) in (6-22), we see that

$$g_M = \frac{(1 + N/JG_1)\,(1 + N/JG_0)}{\frac{N}{J}\left(\frac{1}{G_1} + \frac{1}{G_0}\right) + \left(\frac{N}{J}\right)^2 \frac{1}{G_1 G_0} + \frac{\det \hat{r}_{M+1}}{\det \hat{r}_M}} \tag{6-27}$$

In the limit of large J/N ratios

$$\lim_{J/N \to \infty} g_M = \left[\frac{\det \hat{\underline{r}}_M}{\det \hat{\underline{r}}_{M+1}}\right]_{N=0} = \left[\frac{\det \underline{r}_M}{\det \underline{r}_{M+1}}\right]_{N=0} \tag{6-28}$$

6.3 CALCULATIONS FOR A LINEAR ARRAY

A linear uniformly spaced SLC array is shown in Figure 6-1. The antenna locations are numbered 0,1,2,...M corresponding to the main beam and the M successive SLC antennas. While the antenna spacing along the array is d units, we note that the projected antenna spacing is $d \cos \theta$. Let $R(x)$ denote the correlation coefficient for the received signal fluctuations (for carrier transmission) for two points in space separated by x units and lying in a plane perpendicular to the direction of propagation. Then we readily see that

$$\rho_{mp} = R(d[m-p] \cos\theta) = R_{m-p} \tag{6-29}$$

Assuming the case of saturated ionospheric scintillations, Wittwer [6.1] shows that the spatial correlation function $R(\cdot)$ is closely given by

$$R(x) = \exp - \frac{x^2}{\ell_0^2} \tag{6-30}$$

where ℓ_0 is a correlation "distance". The rms value of the angle of arrival fluctuations σ_θ may be related to ℓ_0 by the following equation [Ref. 6.1]

$$\sigma_\theta = \frac{1}{2\pi} \cdot \frac{\lambda}{\ell_0} \sqrt{2} \tag{6-31}$$

so that (6-30) may be expressed in the alternate form

$$R(x) = \exp \left(- \left[\frac{\sqrt{2}\, \pi x}{\lambda} \sigma_\theta \right]^2 \right) \tag{6-32}$$

and R_m takes the form

$$R_m = \exp\left(-m^2 \left[\frac{\sqrt{2}\,\pi d \cos\theta}{\lambda} \quad \sigma_\theta\right]^2\right) = \alpha^{m^2} \qquad (6\text{-}33)$$

where

$$\alpha = \exp\left(-\left[\frac{\sqrt{2}\,\pi d \cos\theta}{\lambda} \quad \sigma_\theta\right]^2\right) = \exp\left(-\left[\frac{d \cos\theta}{\ell_0}\right]^2\right) \qquad (6\text{-}34)$$

is the correlation coefficient between adjacent antennas.

For the case of large J/N ratios we may use (6-33) in (6-28) to compute the AJ processing gain for M SLC *antennas*

$$g_M = \frac{\det \begin{bmatrix} 1 & \alpha & \alpha^4 & \cdots & \alpha^{(M-1)^2} \\ \alpha & 1 & \alpha & \cdots & \alpha^{(M-2)^2} \\ \vdots & & & & \\ \alpha^{(M-1)^2} & & & \cdots & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & \alpha & \alpha^4 & \cdots & \alpha^{M^2} \\ \alpha & 1 & \alpha & \cdots & \alpha^{(M-1)^2} \\ \vdots & & & & \\ \alpha^{M^2} & & & \cdots & 1 \end{bmatrix}} \qquad (6\text{-}35)$$

Expressions for g_M for M = 1,2,3 are given below

$$g_1 = \frac{1}{1 - \alpha^2} \tag{6-36}$$

$$g_2 = \frac{1 - \alpha^2}{1 - 2\alpha^2 + 2\alpha^6 - 2\alpha^8} = \frac{1}{(1 - \alpha^2)(1 - \alpha^4)} \tag{6-37}$$

$$g_3 = \frac{1 - 2\alpha^2 + 2\alpha^6 - \alpha^8}{1 - 3\alpha^2 + \alpha^4 + 4\alpha^6 - 2\alpha^8 - 2\alpha^{10} - 2\alpha^{12} + 4\alpha^{14} + \alpha^{16} - 3\alpha^{18} + \alpha^{20}}$$

$$= \frac{1}{(1 - \alpha^2)(1 - \alpha^4)(1-\alpha^6)} \tag{6-38}$$

It appears, but has not been proven that

$$g_M = \frac{1}{(1 - \alpha^2)(1 - \alpha^4) \ldots (1-\alpha^{2M})} \tag{6-39}$$

The behavior of g_M for α near 1 is of particular interest because it describes the onset of degradation. If we let

$$\alpha = 1 - \varepsilon \tag{6-40}$$

in (6-39), it is found that

$$g_M \rightarrow \frac{\varepsilon^{-M}}{2^M \, M!} \qquad \text{for } \varepsilon << 1 \tag{6-41}$$

From (6-34) we see that

$$\varepsilon = 1 - \exp\left(- \left[\frac{d \cos\theta}{\ell_0}\right]^2\right) \tag{6-42}$$

When ϵ is small we may use the first two terms in a Taylor expansion of the exponential in (6-42)

$$\epsilon \approx \left[\frac{d \cos\theta}{\ell_0}\right]^2 = \left[\frac{\sqrt{2}\ \pi d \cos\theta}{\lambda}\ \sigma_\theta\right]^2 \qquad (6\text{-}43)$$

It follows that the approximation (6-41) may be expressed as

$$g_M = \frac{\left[\frac{\sqrt{2}\ \pi d \cos\theta}{\lambda}\ \sigma_\theta\right]^{-2M}}{2^M\ M!} \quad ; \quad M = 1,2,3 \qquad (6\text{-}44)$$

Figure 6-2 presents plots relating g_1, g_2 and g_3 in dB to a "decor relation" parameter

$$y = \frac{d \cos\theta}{\ell_0} = \frac{\sqrt{2}\ \pi d \cos\theta}{\lambda}\ \sigma_\theta \qquad (6\text{-}45)$$

using the exact high J/N ratio expressions (6-36) through (6-38) and the Gaussian shaped spatial correlation function (6-30). The dashed lines show the small decorrelation approximation (6-44). Note that these asymptotic approximations which are straight lines on a log-log plot are a good engineering approximation for values of the decorrelation parameter y as large as 0.3 to 0.4 .

The improvement in performance with increasing number of antennas may be understood physically by noting that additional antennas allow the null to be broadened and thus reduce the impact of the angle of arrival fluctuations.

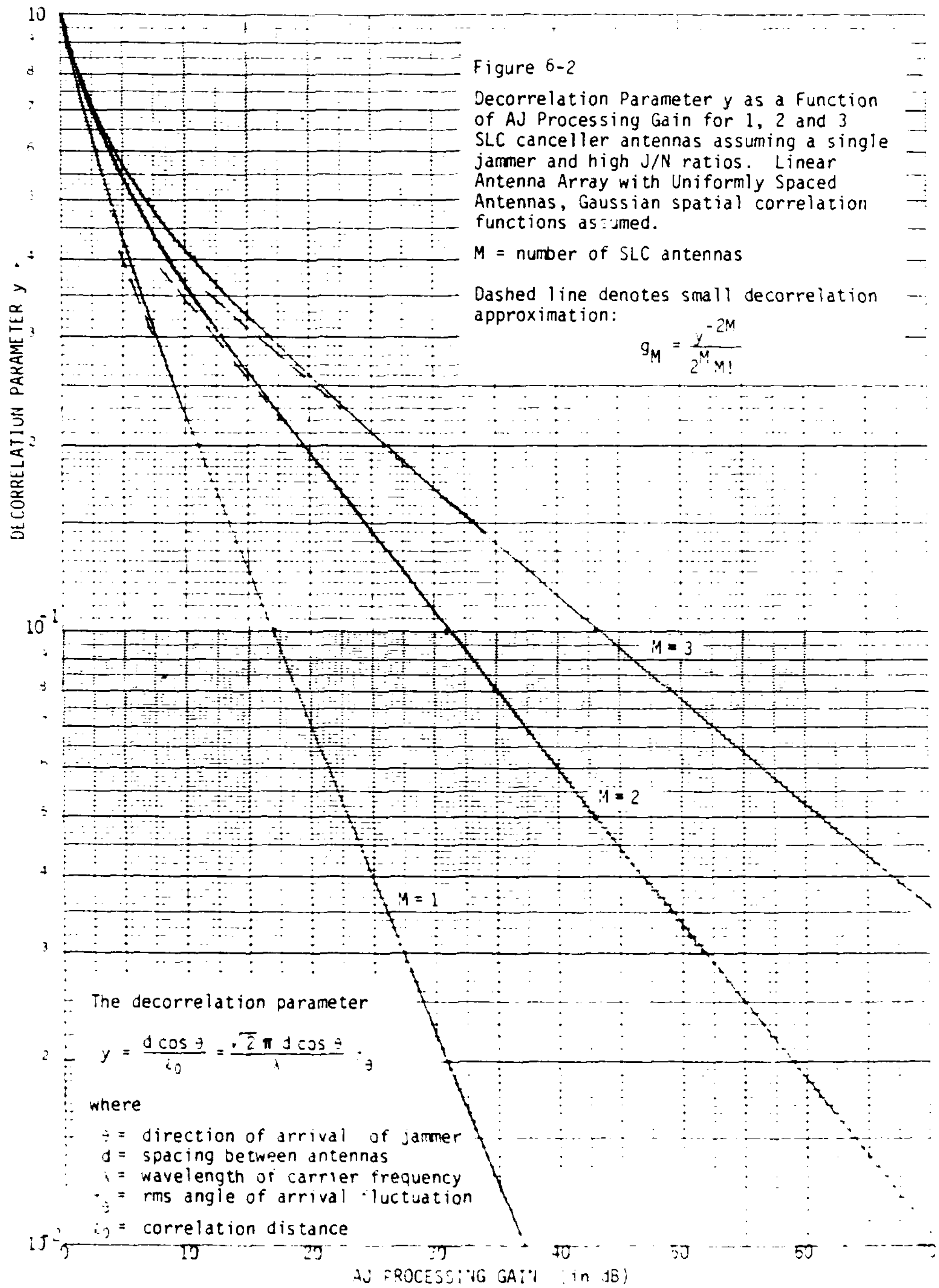

Figure 6-2

Decorrelation Parameter y as a Function of AJ Processing Gain for 1, 2 and 3 SLC canceller antennas assuming a single jammer and high J/N ratios. Linear Antenna Array with Uniformly Spaced Antennas, Gaussian spatial correlation functions assumed.

M = number of SLC antennas

Dashed line denotes small decorrelation approximation:

$$g_M = \frac{y^{-2M}}{2^M M!}$$

It is of interest to examine the anomalous case in which the spatial correlatiton function has the form

$$R(x) = \exp\left(- \frac{|x|}{\ell_0}\right) \tag{6-46}$$

and thus

$$R_m = \beta^{|m|} \tag{6-47}$$

where

$$\beta = \exp\left(- \frac{d \cos\theta}{\ell_0}\right) \tag{6-48}$$

The processing gain is given by the ratio

$$g_M = \frac{\det \begin{bmatrix} 1 & \beta & \cdots & \beta^{M-1} \\ \beta & & & \\ \vdots & & & \\ \beta^{M-1} & & \cdots & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & \beta & \cdots & \beta^{M} \\ \beta & & & \\ \vdots & & & \\ \beta^{M} & & \cdots & 1 \end{bmatrix}} = \frac{1}{1 - \alpha^2} \tag{6-49}$$

Note that this result is independent of M in contrast with the previous example and in apparent disagreement with our heuristic arguments on the benefits of more antennas. However, one may argue that the reasoning is not wrong in this case but ambiguous because the correlation function (6-40) corresponds to the non-physical case of an infinite value for the rms angle of arrival fluctuations. Clearly no null broadening can counteract an infinite rms angle of arrival.

SECTION VII

RESULTS FOR THE THREE-DIMENSIONAL SLC ARRAY WITH MULTIPLE JAMMERS

7.1 INTRODUCTION

This section will formally state the equations that need to be programmed to compute the processing gain of a specified three-dimensional SLC array when subjected to L jammers arriving from specified directions.

7.2 FORMULATION

The antennas are assumed arranged in a uniformly spaced three-dimensional grid pattern with grids parallel to the x, y, and z axes. Thus the location of an arbitrarily selected antenna can be specified by a radius vector

$$\underline{r}_{mpq} = \underline{i}md_x + \underline{j}pd_y + \underline{k}qd_z \tag{7-1}$$

where $\underline{i}$, $\underline{j}$, $\underline{k}$ are unit vectors along the x, y, and z directions; d_x, d_y, and d_z are the uniform spacing between antenna elements aligned parallel to the x, y, and z directions, respectively; and m, p, and q are integers. By letting the antenna degenerate, a planar and simple linear array can be modeled. Thus the formulation includes the two- and one-dimensional SLC performances as special cases. We assume that m, p, and q range over the values $(-M_-, M_+)$, $(-P_-, P_+)$ and $(-Q_-,Q_+)$ respectively.

It is convenient to specify an antenna and its output by the subscript mpq. We assume that the main beam antenna is located at the origin of coordinates. Thus, the output of the main beam antenna is given by

$$w_{000}(t) = s(t)\ h_{0,000}(t) + j_1(t)\ h_{1,000}(t) + j_L(t)\ h_{L,000}(t) + n_{000}(t)$$

while the output of the mpq antenna is

$$w_{mpq}(t) = j_1(t)\ h_{1,mpq}(t) + j_2(t)\ h_{2,mpq}(t) + \dots + j_L(t)\ h_{L,mpq}(t) + n_{mpq}(t) \tag{7-2}$$

where $j_\ell(t)$ is the ℓ^{th} jammer, $h_{\ell,mpq}(t)$ is the complex time variant gain characterizing transmission from the ℓ^{th} transmitter location to the mpq antenna output, and $n_{mpq}(t)$ is the thermal noise component at the mpq antenna output. Note that we have assumed no signal components in any of the SLC antenna outputs, i.e.,

$$h_{0,mpq} = 0 \qquad ; \qquad mpq \neq 000 \tag{7-3}$$

The complex transmission gain $h_{\ell,mpq}$ can be factored into the following form

$$h_{\ell,mpq}(t) = A_{mpq}(\theta_\ell,\phi_\ell)\; e^{j\gamma_{mpq}(\theta_\ell,\phi_\ell)}\; g_{\ell,mpq}(t) \tag{7-4}$$

where θ_ℓ,ϕ_ℓ are the spherical angular coordinates characterizing the direction of arrival of the ℓ^{th} jammer, $A_{mpq}(\theta,\phi)$ is the complex antenna gain pattern of the mpq antenna, $\gamma_{mpq}(\theta_\ell,\phi_\ell)$ is the phase delay of the signal arriving at the mpq antenna from the ℓ^{th} jammer relative to the phase delay at the 000 position [$\gamma_{000}(\theta_\ell,\phi_\ell)$ is set = 0] and $g_{\ell,mpq}(t)$ is the complex time-variant channel gain characterizing the path from the ℓ^{th} jammer to the mpq antenna.

To simplify the antenna model, a default option should allow all the SLC antennas to have the same pattern, i.e.,

$$A_{mpq}(\theta,\phi) = B(\theta,\phi) \qquad ; \qquad mpq \neq 000 \tag{7-5}$$

while

$$A_{000}(\theta,\phi) = A(\theta,\phi) \tag{7-6}$$

To simplify subsequent notation we assume temporarily that a one-one mapping has been established between the triplet values

$$\{mpq\ ;\ m = -M_-, \ldots -1,0,1,\ldots M_+,\quad p = -P_-, \ldots -1,0,1,\ldots P_+,\quad q = -Q_-, \ldots -1,0,1,\ldots Q_+\ \}$$

and the integers $\{n;n=0,1, \ldots K\}$, the triplet corresponding to the integer n being denoted by $m_n p_n q_n$. We assign n=0 to the triplet 0,0,0 denoting the main beam.

Thus, rewriting Eqs.(7-1) through (7-4),

$$w_0(t) = s(t)\ h_{00}(t) + j_1(t)\ h_{10}(t) + \ldots + j_L(t)\ h_{L0}(t) + n_0(t) \tag{7-7}$$

$$w_n(t) = j(t)\ h_{1n}(t) + j_2(t)\ h_{2n}(t) + \ldots + j_L(t)\ h_{Ln}(t) + n_n(t) \tag{7-8}$$

$$h_{0n} = 0 \quad ; \quad n \neq 0 \tag{7-9}$$

$$h_{\ell n}(t) = A_n(\theta_\ell,\phi_\ell)\ e^{j\gamma_n(\theta_\ell,\phi_\ell)}\ g_{\ell n}(t) \tag{7-10}$$

The SNR in the main beam, i.e., the SNR without adaptation, is

$$\rho_0 = \frac{G_{00}\ S}{\sum_{\ell=1}^{L} J_\ell\ G_{\ell 0} + N} \tag{7-11}$$

where we have used the notation

$$G_{\ell n} = \left|\ A_n(\theta_\ell,\phi_\ell)\ \right|^2 \tag{7-12}$$

for the n^{th} antenna gain in the direction of the ℓ^{th} jammer and we have used the normalization

$$\overline{|g_{\ell n}|^2} = 1 \tag{7-13}$$

The SNR after subtraction of the SLC antenna outputs, i.e., the SNR after adaptation, is given by

$$\rho = \frac{G_{00}\, S}{\sum_{\ell=1}^{L} J_\ell\, G_{\ell 0} + N - \underline{R}^H\, \underline{C}^{-1}\, \underline{R}} \tag{7-14}$$

so that the processing gain is given by

$$g = \frac{\sum_{\ell=1}^{L} J_\ell\, G_{\ell 0} + N}{\sum_{\ell=1}^{L} J_\ell\, G_{\ell 0} + N - \underline{R}^H\, \underline{C}^{-1}\, \underline{R}} \tag{7-15}$$

where

$$\underline{R} = \{\, \overline{w_0 w_n^*} \;; \quad n = 1,2,\ldots,K \,\} \tag{7-16}$$

$$\underline{C} = \{\, \overline{w_n w_r^*} \;; \quad n,r = 1,2,\ldots,K \,\} \tag{7-17}$$

Using Eqs.(7-7) through (7-10) we can express $\underline{R}$ and $\underline{C}$ as sums of matrices each dependent upon a single jammer (assuming uncorrelated jammers).

$$\underline{R} = \sum_{\ell=1}^{L} J_\ell\, A_0(\theta_\ell,\phi_\ell)\, \{\, A_n^*(\theta_\ell,\phi_\ell)\, e^{-j\gamma_n(\theta_\ell,\phi_\ell)}\, \rho_{\ell n0} \;; n = 1,2,\ldots,K \,\}$$

$$\underline{C} = \sum_{\ell=1}^{L} \{\, J_\ell\, A_r(\theta_\ell,\phi_\ell)\, A_n^*(\theta_\ell,\phi_\ell)\, e^{-j\,[\gamma_r(\theta_\ell,\phi_\ell)-\gamma_n(\theta_\ell,\phi_\ell)]}\, \rho_{\ell nr} + N\delta_{nr} \;; n,r = 1,2,\ldots,K \,\} \tag{7-18}$$

where

$$\delta_{nr} = \begin{cases} 1 & ; \quad n = r \\ 0 & ; \quad n \neq r \end{cases} \tag{7-19}$$

and

$$\rho_{\ell nr} = \overline{g^*_{\ell n}\, g_{\ell r}} \tag{7-20}$$

is the spatial correlation coefficient between reception at the n^{th} and r^{th} antennas corresponding to transmission from the position of the ℓ^{th} jammer.

Assuming our interest is in SNR or processing gain we can normalize both S and J_n to N, i.e., set N=1 in the above expressions and regard J_n as the J_n/N ratio.

It remains to specify the correlation coefficients $\rho_{\ell nr}$ and the phase delays $\gamma_n(\theta_\ell,\phi_\ell)$. We will develop expressions for these terms as a function of the antenna element positions, the arrival angles of the ℓ^{th} jammer (θ_ℓ,ϕ_ℓ), and a specified perpendicular correlation function $R_\perp(x)$. The latter is the spatial correlation function in a plane perpendicular to the direction of propagation of a received signal.

As discussed earlier, in the computation of the spatial correlation coefficients, it is sufficient to use modified antenna element positions obtained by projecting the antenna positions on a plane perpendicular to the direction of propagation of the jammer. Thus if $d_{\ell nr}$ denotes the distance between the n^{th} and r^{th} <u>projected</u> antennas for ℓ^{th} jammer direction of arrival,

$$\rho_{\ell nr} = R_\perp(d_{\ell nr}) \tag{7-21}$$

Then for computation of $\rho_{\ell nr}$ it remains only to obtain an expression for $d_{\ell nr}$ once $R_{\perp}(\cdot)$ is specified.

Consider now the phase delay term $\gamma_n(\theta_\ell,\phi_\ell)$. Assuming that the jammer is in the far zone of the antenna $\gamma_n(\theta_\ell,\phi_\ell)$ depends only on modified antenna positions obtained by projecting the actual antenna positions on a line from the ℓ^{th} jammer to the main beam antenna. Let the position of the main beam antenna be the origin of coordinates along the line and $s_{\ell n}$ be the position of the n^{th} projected antenna, then

$$\gamma_n(\theta_\ell,\phi_\ell) = 2\pi \frac{s_{\ell n}}{\lambda} \tag{7-22}$$

where λ is the wavelength.

The distances $d_{\ell nr}, s_{\ell n}$ may be obtained from the antenna positions in the x,y,z coordinate system by means of a rotation of coordinates. Figure 1 shows the relation between the x,y,z coordinate system and the x^ℓ,y^ℓ,z^ℓ coordinate system. The z^ℓ axis coincides with a line from the origin of x,y,z system (the main beam antenna location) to the ℓ^{th} jammer. This line has spherical angular coordinates (θ_ℓ,ϕ_ℓ). The y^ℓ axis is perpendicular to z^ℓ and in the plane containing the z and z^ℓ axes. Since the x^ℓ,y^ℓ,z^ℓ coordinate system is an orthogonal system the $x^\ell y^\ell$ plane is perpendicular to the z^ℓ axis. It follows from our construction that the $x^\ell y^\ell$ coordinates of the antennae are just those of the plane-projected antennas needed to compute $d_{\ell nr}$ while the z^ℓ coordinate provides $s_{\ell n}$. Specifically if $(x_n^\ell,y_n^\ell,z_n^\ell)$ denote the coordinates of the n^{th} antenna in the (x^ℓ,y^ℓ,z^ℓ) coordinate system,

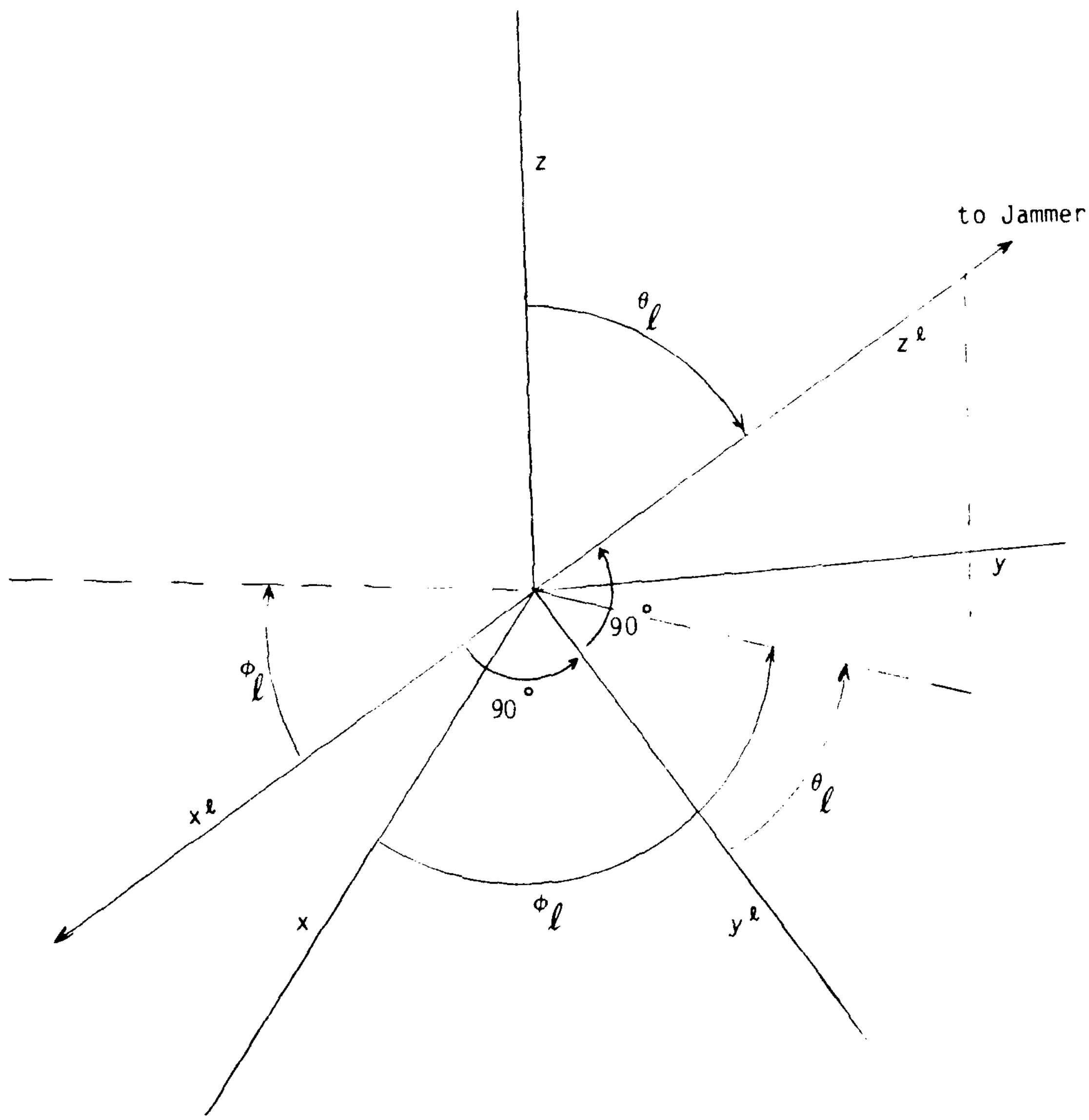

Figure 7-1 Relation Between x,y,z and x^ℓ,y^ℓ,y^ℓ Coordinate System

$$d_{\ell nr} = \sqrt{(x_n^\ell - x_r^\ell)^2 + (y_n^\ell - y_r^\ell)^2} \qquad (7\text{-}23)$$

$$s_{\ell n} = z_n^\ell \qquad (7\text{-}24)$$

The relationship between $(x_n^\ell, y_n^\ell, z_n^\ell)$ and (x_n, y_n, z_n) is given by

$$x_n^\ell = x_n \sin\phi_\ell - y_n \cos\phi_\ell \qquad (7\text{-}25)$$

$$y_n^\ell = x_n \cos\theta_\ell \cos\phi_\ell + y_n \cos\theta_\ell \sin\phi_\ell - z_n \sin\theta_\ell \qquad (7\text{-}26)$$

$$z_n^\ell = x_n \sin\theta_\ell \cos\phi_\ell + y_n \sin\theta_\ell \sin\phi_\ell + z_n \cos\theta_\ell \qquad (7\text{-}27)$$

From (7-1) we note that

$$x_n = m_n d_x \qquad (7\text{-}28)$$

$$y_n = p_n d_y \qquad (7\text{-}29)$$

$$z_n = q_n d_z \qquad (7\text{-}30)$$

where m_n, p_n, q_n is the triplet corresponding to the n^{th} antenna in our one-one mapping. Using (7-25) through (7-30) in (7-23) and (7-24) we see that

$$d^2_{\ell nr} = \left[d_x(m_n - m_r) \sin\phi_\ell - d_y(p_n - p_r) \cos\phi_\ell\right]^2$$

$$+ \left[d_x(m_n - m_r) \cos\theta_\ell \cos\phi_\ell + d_y(p_n - p_r) \cos\theta_\ell \sin\phi_\ell - d_z(q_n - q_r) \sin\theta_\ell\right]^2 \qquad (7\text{-}31)$$

$$s_{\ell n} = m_n d_x \sin\theta_\ell \cos\phi_\ell + p_n d_y \sin\theta_\ell \sin\phi_\ell + q_n d_z \cos\theta_\ell \qquad (7\text{-}32)$$

As discussed earlier, a suitable perpendicular correlation function is given by

$$R_{\perp}(x) = e^{-x^2/\ell_0^2} \tag{7-33}$$

where ℓ_0 is a correlation distance parameter. This parameter is related to σ_θ, the rms value of angle of arrival fluctions, by

$$\sigma_\theta = \frac{1}{\sqrt{2}\,\pi} \frac{\lambda}{\ell_0} \tag{7-34}$$

The above formulae allow a computation of processing gain for a three-dimensional array subjected to noise jamming from L independent sources arriving at angles (θ_ℓ, ϕ_ℓ) ; $\ell = 1,2,\ldots L$.

SECTION VIII

ERROR RATES WITH A FADING NON-ADAPTIVE SLC ARRAY

8.1 INTRODUCTION

In this section we will develop expressions for the error rate of modems used in conjunction with an SLC (sidelobe canceller) adaptive array. It is assumed that saturated ionospheric scintillations nave occurred, causing the propagation medium to produce complex Gaussian fluctuations on received signals. It is also assumed that the adaptive array time constants are too slow to adapt to the fading and that the signal bandwith is narrow enough to avoid significant frequency selective fading. Thus the performance degradation suffered by the array is caused by spatial selectivity and non-adaptation to the fading. The subsequent analysis will be concerned with the error rate performance of a digital modem connected to the output after the SLC has attempted to cancel the jamming signals. As in the case of the adaptive array, two cases may be considered: the modem time constants may or may not allow adaptation to the fading. In this section we consider the case of a fading adaptive modem. In order for adequate error rate performance to result it is necessary to employ coding, interleaving and forward error correction techniques. The raw error rates computed here may be used to evaluate the improvement offered by coding techniques. Section 8.7 presents numerical results on error for the simplest case - one SLC antenna and one jammer.

8.2 SYSTEM DESCRIPTION

The system to be analyzed is shown in Figure 8-1. An array of M auxiliary antennas with adaptive complex weights and a combiner form a side-lobe canceller (SLC) which by subtraction attempts to cancel the L jamming signals appearing in the sidelobes of the signal or main beam antenna. The "cancelled" output is fed both to a digital modem for extraction of the output data stream and to a device which implements the adaptation algorithm. This algorithm attempts to set the complex weights p_1, p_2, ..., p_M for minimum output power. However, the adaptation process is assumed to be too slow to follow the fading.

The combiner output c(t) can be expressed as

$$c(t) = s(t)\, h_{00}(t) + \sum_{\ell=1}^{L} i_\ell(t)\, h_{\ell 0}(t) + n_0(t) - \sum_{m=1}^{M} p_m \left(\sum_{\ell=1}^{L} i_\ell(t)\, h_{\ell m}(t) + n_m(t) \right) \qquad (8\text{-}1)$$

where s(t) is the transmitted signal, $\{i_\ell(t);\ \ell=1,2,\ldots,L\}$ is the set of L jamming signals, n_m is the thermal noise in the m^{th} antenna output, and $h_{\ell m}(t)$ is a complex time variant multiplier characterizing transmission from the ℓ^{th} jammer to the m^{th} antenna output. We may rewrite (8-1) in the form

$$c(t) = s(t)\, h_0(t) + \sum_{\ell=1}^{L} i_\ell(t)\, h_\ell(t) + n(t) \qquad (8\text{-}2)$$

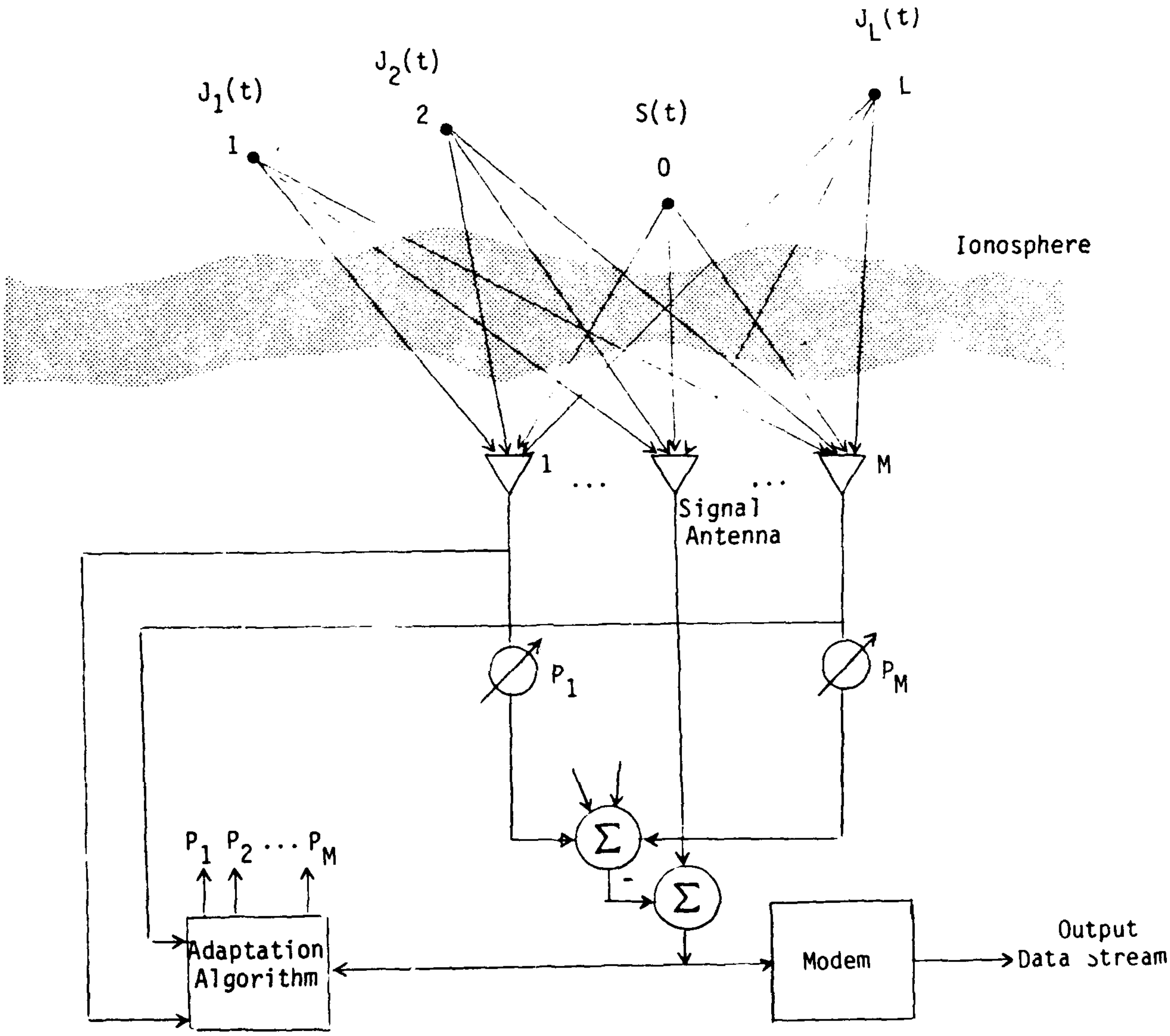

Figure 8-1 System Block Diagram.

where

$$h_\ell(t) = h_{\ell 0}(t) - \sum_{m=1}^{M} p_m \, h_{\ell m}(t) \tag{8-3}$$

$$\eta(t) = \eta_0(t) - \sum_{m=1}^{M} p_m \, \eta_m(t) \tag{8-4}$$

$$h_{00}(t) = h_0(t) \tag{8-5}$$

Note that $i_\ell(t)\, h_\ell(t)$ is the residual jamming component after cancellation due to the ℓ^{th} jammer, while $\eta(t)$ is the output thermal noise.

The transfer functions $h_{\ell m}(t)$ may be expressed in the form

$$h_{\ell m}(t) = A_m(\theta_\ell,\phi_\ell)\, e^{j\gamma_m(\theta_\ell,\phi_\ell)}\, g_{\ell m}(t) \tag{8-6}$$

where (θ_ℓ,ϕ_ℓ) are the angular spherical coordinates of the ℓ^{th} jammer, $\gamma_m(\theta_\ell,\phi_\ell)$ is the phase delay of the ℓ^{th} received jammer signal arriving at the m^{th} antenna measured relative to the signal antenna, $A_m(\theta,\phi)$ is the far field pattern of the m^{th} antenna, and $g_{\ell m}(t)$ is the complex time variant channel characterizing the path from the ℓ^{th} jammer to the m^{th} antenna. The latter has been normalized so that

$$\overline{|g_{\ell m}(t)|^2} = 1 \tag{8-7}$$

and relative phase shifts due to path delay difference have been removed and incorporated in the phase shifts $\{\gamma_m(\theta_\ell,\phi_\ell)\ ;\ m=1,\ldots,M\}$. When the jammer and signal transmitters are far enough apart and saturated fading occurs, it may be assumed that the path gains are statistically independent complex Gaussian variables.

The performance of coherent, differentially coherent, and incoherent, binary and quaternary modems will be evaluated. In all cases it is assumed that the error rate for such modems with a non-fading non-jammed channel may be expressed in the form

$$P_b = f(\rho) \tag{8-8}$$

where ρ is the ratio of the energy/bit to the noise power density. We further assume that prior to detection or decision operations in the modem, the processing may be characterized as complex linear filtering. The jammers are assumed to have spectra flat over the bandwidth of this linear filtering and to have filter outputs characterizable as independent complex Gaussian processes when channel gains are fixed.

As a consequence of the above assumptions the modem error rate <u>conditioned on the channel complex gains</u> is given by

$$P_e = f\left(\frac{E_b}{N_e}\right) \tag{8-9}$$

where E_b is the energy bit and N_e is an equivalent noise power density

$$N_e = \sum_{\ell=1}^{L} N_\ell |h_\ell|^2 + N_0\left[1 + \sum_{m=1}^{M} |p_m|^2\right] \tag{8-10}$$

in which N_ℓ is the one-sided power spectral density of the real ℓ^{th} jamming process in the vicinity of the signal carrier frequency, and N_0 is the common one-sided power spectral density of the real thermal noises at the antenna outputs. E_b may be expressed in the form

$$E_b = \frac{1}{2b}\int_0^T \overline{|s^2(t)|}\, dt\, |h_0|^2 = \frac{ST}{2b}\, \overline{|h_0|^2} \qquad (8\text{-}11)$$

where S is the transmitted power, T is the symbol duration, and b is the number of bits/symbol.

We define

$$s_0 = \frac{ST}{2bN_0}\, \overline{|h_0|^2} = \frac{S}{N}\, \frac{TW}{2b}\, \overline{|h_0|^2} \qquad (8\text{-}12)$$

as the average signal energy/bit/thermal noise power density in the main beam (W is the receiver bandwith and N is the thermal noise power)., and

$$n_\ell = \frac{N_\ell \overline{|h_\ell|^2}}{N_0} = \frac{J_\ell}{N}\, \overline{|h_\ell|^2} \qquad (8\text{-}13)$$

as the ratio of the output jammer power due to the ℓ^{th} jammer to the thermal noise power where J_ℓ is the ℓ^{th} jammer transmitter power. Equivalently, n_ℓ is the ratio of average received jammer power density to thermal noise for the ℓ^{th} jammer. With these definitions

$$\rho = \frac{s_0 |z_0|^2}{\sum_{\ell=1}^{L} n_\ell |z_\ell|^2 + p} \qquad (8\text{-}14)$$

where

$$p = 1 + \sum_{m=1}^{M} |p_m|^2 \qquad (8\text{-}15)$$

and we have defined the normalized variables

$$z_\ell = \frac{h_\ell}{\sqrt{\overline{|h_\ell|^2}}} \quad ; \quad \ell = 1,2,\ldots L \tag{8-16}$$

8.3 ERROR RATE ANALYSIS FOR A SINGLE JAMMER: INDEPENDENTLY FLUCTUATING SIGNAL AND JAMMER CHANNELS

The modem error rate may be evaluated by averaging the conditional error rate (with the channel gains frozen) over the channel fluctuations, i.e.,

$$P_e = \int_0^\infty f(\rho)\, W(\rho)\, d\rho \tag{8-17}$$

where $W(\rho)$ is the probability density function of ρ in Eq.(8-15). We consider now the evaluation of $W(\rho)$.

$W(x)$ may be obtained as the derivative of the probability distribution function

$$W(x) = \frac{d}{dx} F(x) \tag{8-18}$$

where

$$F(x) = \Pr(\rho \le x) \tag{8-19}$$

Using (8-15),

$$\Pr(\rho \le x) = \Pr\left(s_0 |z_0|^2 - x \sum_{\ell=1}^{L} n_\ell |z_\ell|^2 \le px\right) \tag{8-20}$$

We consider in this section the use of a single jammer and independent fluctuations for the signal and jammer channels, i.e., h_0 and h_1 are assumed statistically independent. When the jammer and communicator are separated sufficiently, the statistical independence will be valid. In this case we need tc compute the canonic probability

$$Pr(q \leq Q) = Pr(r_0 - r_1 \leq Q) \tag{8-21}$$

where

$$r_k = |\mu_k|^2 \quad ; \quad k=0,1 \tag{8-22}$$

and $\{\mu_k \; ; \; k=0,1\}$ are independent complex Gaussian variables. The probability density function of r_k is given by

$$W_k(r) = \frac{1}{a_k} \exp(- \frac{r}{a_k}) \; U(s) \quad ; \quad k=0,1 \tag{8-23}$$

where

$$a_k = \overline{r_k} = \overline{|\mu_k|^2} \tag{8-24}$$

and

$$U(r) = \begin{cases} 1 & ; \quad r \geq 0 \\ 0 & ; \quad r < 0 \end{cases} \tag{8-25}$$

The density function of q is just the convolution of the density function of r_0 with that of $-r_1$ or

$$w(q) = \int W_0(s) \; W_1(s-q) \; ds \tag{8-26}$$

Carrying out (8-26),

$$W(q) = \begin{cases} \dfrac{e^{-q/a_0}}{a_0 + a_1} & ; \quad q > 0 \\ \\ \dfrac{e^{q/a_1}}{a_0 + a_1} & ; \quad q < 0 \end{cases} \tag{8-27}$$

Since

$$\Pr(q \leq Q) = \int_{-\infty}^{Q} W(q)\, dq \tag{8-28}$$

we find that

$$\Pr(q \leq Q) = \begin{cases} \dfrac{a_1\, e^{Q/a_1}}{a_1 + a_0} & ; \quad Q < 0 \\[2ex] 1 - \dfrac{a_0\, e^{-Q/a_0}}{a_1 + a_0} & ; \quad Q \geq 0 \end{cases} \tag{8-29}$$

Upon identifying (for L=1)

$$a_0 = s_0 \tag{8-30}$$

$$a_1 = xn \qquad (n_1 \equiv n) \tag{8-31}$$

$$Q = px \tag{8-32}$$

in (8-29), we see that

$$F(x) = \Pr(\rho \leq x) = 1 - \frac{s_0}{s_0 + xn}\, e^{-px/s_0}; \; x > 0 \tag{8-33}$$

or, defining the ratios

$$P = p/s_0 \tag{8-34}$$

$$R = n/s_0 \tag{8-35}$$

$$F(x) = 1 - \frac{1}{1 + xR}\, e^{-Px} \tag{8-36}$$

The density function of ρ is obtained by differentiating (8-36) [see (8-18)],

$$W(x) = \left[\frac{P}{1 + xR} + \left(\frac{R}{1 + xR} \right)^2 \right] e^{-Px} \quad ; \; x \geq 0 \qquad (8\text{-}37)$$

Using (8-17), the following integral yields the error probability for the single jammer case:

$$P_e = \int_0^\infty f(\rho) \left[\frac{P}{1 + \rho R} + \left(\frac{R}{1 + \rho R} \right)^2 \right] e^{-P\rho} \, d\rho \qquad (8\text{-}38)$$

An alternate expression, which is sometimes more convenient, may be obtained from (8-38) by integrating by parts,

$$P_e = \int_0^\infty \left[1 - \frac{1}{1 + \rho R} e^{-P\rho} \right] \left[-\dot{f}(\rho) \right] d\rho \qquad (8\text{-}39)$$

The following represent well-known expressions for $f(\rho)$,

$$f(\rho) = \frac{1}{2} \exp\left(-\frac{\rho}{2}\right) \quad ; \quad \text{incoherent FSK} \qquad (8\text{-}40)$$

$$f(\rho) = \frac{1}{2} e^{-\rho} \quad ; \quad \text{binary and quaternary DPSK} \qquad (8\text{-}41)$$

$$f(\rho) = Q(\sqrt{2\rho}) \quad ; \quad \text{binary and quaternary PSK} \qquad (8\text{-}42)$$

$$f(\rho) = Q(\sqrt{\rho}) \quad ; \quad \text{binary coherent FSK} \qquad (8\text{-}43)$$

where FSK, DPSK, and PSK denote binary frequency shift keying, differential phase shift keying, and phase shift keying, respectively, and

$$Q\left(\sqrt{2\rho}\right) = \int_{\sqrt{2\rho}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\gamma^2/2} d\gamma \qquad (8\text{-}44)$$

Noting that

$$-\dot{f}(\rho) = \begin{cases} \frac{1}{4} \exp\left(-\frac{\rho}{2}\right) & ; \quad \text{incoherent FSK} \qquad (8\text{-}45) \\ \frac{1}{2} \exp(-\rho) & ; \quad \text{binary and quaternary DPSK} \qquad (8\text{-}46) \end{cases}$$

$$-\dot{f}(\rho) = \begin{cases} \frac{1}{\sqrt{2\rho}} \frac{1}{\sqrt{2\pi}} \exp(-\rho) & ; \quad \text{binary and quaternary PSK} \qquad (8\text{-}47) \\ \frac{1}{2\sqrt{\rho}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\rho}{2}\right) & ; \quad \text{binary coherent FSK} \qquad (8\text{-}48) \end{cases}$$

and using (8-47) in (8-39), we find that

$$P_e = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{2\rho}} \exp(-\rho) \left[1 - \frac{1}{1 + \rho R} e^{-P\rho}\right] d\rho \qquad (8\text{-}49)$$

Using the change of variable $y = \sqrt{2\rho}$,

$$P_e = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \left[1 - \frac{1}{1+y^2R/2} \exp(-Py^2/2)\right] dy \qquad (8\text{-}50)$$

The first term may be integrated, yielding

$$P_e = \frac{1}{2} - \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \frac{\exp(-(1+P)y^2/2)}{1+y^2R/2} dy \qquad (8\text{-}51)$$

Upon the change of variable $y\sqrt{R/2} \to z$

$$P_e = \frac{1}{2} - \sqrt{\frac{2}{R}} \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{\exp\left(-\left(\frac{1+P}{R}\right)z^2\right)}{1+z^2}\, dz \quad ; \quad \text{PSK} \qquad (8\text{-}52)$$

More generally, since (see page 314 of Ref.[8.1])

$$\int_0^\infty \frac{\exp(-\lambda x^2)}{1+x^2}\, dx = \pi\, e^{\lambda}\, Q(\sqrt{2\lambda}) \qquad (8\text{-}53)$$

we see that

$$P_e = \frac{1}{2} - \sqrt{\frac{\pi}{R}}\; e^{\left(\frac{1+P}{R}\right)}\; Q\left(\sqrt{2\left(\frac{1+P}{R}\right)}\right) \quad ; \quad \text{PSK} \qquad (8\text{-}54)$$

Using the asymptotic expansion

$$Q(x) = \frac{1}{x\sqrt{2\pi}}\, e^{-x^2/2} \left[1 - \frac{1}{x^2} + \frac{1}{x^4} \cdots \right] \qquad (8\text{-}55)$$

we deduce the approximations

$$\begin{aligned} P_e &\to \frac{1}{2}\left[1 - \frac{1}{\sqrt{1+P}}\left(1 - \frac{R}{2(1+P)}\right)\right] \quad ; \quad R \ll 1+P \\ &\to \frac{R+P}{4} \quad ; \quad R \ll 1,\ P \ll 1 \end{aligned} \qquad (8\text{-}56)$$

In an exactly analogous procedure, using (8-48) in (8-39) we find that

$$P_e = \frac{1}{2} - \sqrt{\frac{\pi}{2R}}\; e^{\frac{1+2P}{2R}}\; Q\left(\sqrt{\frac{1+2P}{R}}\right) \quad ; \quad \text{coherent FSK} \qquad (8\text{-}57)$$

We now consider binary and quaternary DPSK and insert (8-46) in (8-39), resulting in the integral

$$P_e = \int_0^\infty \frac{1}{2}\; e^{-\rho}\left[1 - \frac{1}{1+\rho R}\, e^{-P\rho}\right] d\rho \qquad (8\text{-}58)$$

This integral can be expressed in terms of the exponential integral function

$$E_1(x) = \int_0^\infty \frac{e^{-t}}{t}\, dt \qquad (8\text{-}59)$$

as

$$P_e = \frac{1}{2}\left(1 - \frac{e^{\frac{1+P}{R}}}{R}\, E_1\left(\frac{1+P}{R}\right)\right) \quad ; \quad \text{DPSK} \qquad (8\text{-}60)$$

With the 3 dB difference for incoherent FSK

$$P_e = \frac{1}{2}\left(1 - \frac{1}{2R}\, e^{\frac{1+2P}{2R}}\, E_1\left(\frac{1+2P}{2R}\right)\right) \quad ; \quad \text{incoherent FSK} \qquad (8\text{-}61)$$

Since

$$E_1(z) = \frac{e^{-z}}{z} \left[1 - \frac{1}{z} + \frac{1}{z^2} \cdots \right] \tag{8-62}$$

we find that

$$P_e \to \frac{1}{2} \left[1 - \frac{1}{1+P} \left(1 - \frac{R}{1+P}\right) \right] \quad ; R << 1+P$$

$$\to \frac{R+P}{2} \quad ; R,P << 1 \tag{8-63}$$

8.4 ERROR RATE ANALYSIS FOR MULTIPLE JAMMERS: INDEPENDENT FLUCTUATING SIGNAL AND JAMMER CHANNELS

In this section we consider the evaluation of error rate for multiple jammers. Equations (8-17) through (8-20) are applicable to the multiple jammer case. Computation of (8-20) requires an evaluation of a probability of the generic form,

$$H(Q) \quad = \quad Pr(r_0 - \sum_{\ell=1}^{L} r_\ell \leq Q) \tag{8-64}$$

where

$$r_\ell = \quad |\mu_\ell|^2 \qquad ; \qquad m = 2,1,\ldots M \tag{8-65}$$

and μ_ℓ is a complex Gaussian random variable. In this section we shall assume that the jammers and communicator are mutually separated far enough apart that the set $\{h_\ell(t)\ ;\ \ell=0,1,\ldots L\}$ are statistically independent. Then the set $\{\mu_\ell\ ;\ \ell=0,1,\ldots L\}$ may be taken as statistically independent.

We shall first find the p.d.f. (probability density function) of

$$q \quad = \quad r_0 - \sum_{\ell=1}^{L} r_\ell \tag{8-66}$$

by transforming its c.f. (characteristic function). This is, if

$$C(s) \quad = \quad \overline{e^{-sq}} \quad = \quad \int W(q)\ e^{-sq}\ dq \tag{8-67}$$

is the c.f. of q (we have generalized to a complex argument s), then the p.d.f. of q is given by

$$W(q) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} C(s)\, e^{sq}\, ds \tag{8-68}$$

Equations (8-67) and (8-68) are two-sided Laplace transform pairs. The integral in (8-68) is a contour integral closed around the left half plane for $q > 0$ and around the right half plane for $q < 0$.

Because of the assumed independence of the set $\{r_\ell\}$, the c.f. of q factors as follows

$$C(s) = \overline{e^{-s\left(r_0 - \sum_{\ell=1}^{L} r_\ell\right)}} = C_0(s) \prod_{\ell=1}^{L} C_\ell(-s) \tag{8-69}$$

where

$$C_\ell(s) = \overline{e^{-sr_\ell}} \qquad ; \quad \ell = 0,1,\ldots L \tag{8-70}$$

is the c.f. of r_ℓ. This c.f. is given by [see (8-23) and (8-24)]

$$\begin{aligned} C_\ell(s) &= \int_0^\infty e^{-sx} \frac{1}{a_\ell} \exp\left(- \frac{x}{a_\ell} \right) dx \\ &= \frac{1/a_\ell}{s + 1/a_\ell} \end{aligned} \tag{8-71}$$

Thus using (8-71) in (8-69)

$$C(s) = \frac{1/a_0}{s + 1/a_0} \prod_{\ell=1}^{L} \left(\frac{-1/a_\ell}{s - 1/a_\ell} \right) \tag{8-72}$$

We desire $H(Q)$, (8-64), which is the d.f. (distribution function) of q since from this d.f. the d.f. of ρ, $F(x)$, Eqs.(8-19) and (8-20), may be trivially determined and thereby the desired error rate via the integral

$$P_e = \int_0^\infty F(\rho) \, (-\dot{f}(\rho)) \, d\rho \tag{8-73}$$

where $\dot{f}(\rho)$ is given by Eqs. (8-39) through (8-48).

We may obtain a contour integral representation of $H(Q)$ as follows

$$H(Q) = \int_{-\infty}^{Q} W(q) \, dq = \frac{1}{2\pi j} \oint_{-j\infty}^{j\infty} \frac{C(s)}{s} \, e^{sQ} \, ds \tag{8-74}$$

where in carrying out this integration the path of integration along the j axis is indented to the right of the pole at the origin and the contour closes in a large semi-circle in the left half plane.

Since the counter includes only the poles at $s = -1/a_0$ and $s = 0$, from Cauchy's residue theorem,

$$H(Q) = \mathrm{Res}\left\{\frac{C(s)}{s} e^{sQ}\right\}_{s=-1/a_0} + \mathrm{Res}\left\{\frac{C(s)}{s} e^{sQ}\right\}_{s=0} \tag{8-75}$$

or

$$H(Q) = 1 - \left(\prod_{\ell=1}^{L} \frac{a_0}{a_0 + a_\ell} \right) e^{-Q/a_0} \quad ; \qquad Q \geq 0 \tag{8-76}$$

Upon making the identifications

$$a_0 = s_0 \tag{8-77}$$

$$a_\ell = xn_\ell \quad ; \quad \ell = 1,2,\ldots L \tag{8-78}$$

$$Q = px \tag{8-79}$$

we see from (8-20), (8-64) and (8-65) [note that $|q_\ell|^2 = 1$], that

$$F(x) = 1 - \left(\prod_{\ell=1}^{L} \frac{1}{1 + xR_\ell} \right) e^{-Px} \tag{8-80}$$

where P is given by (8-34) and

$$R_\ell = n_\ell / s_0 \tag{8-81}$$

The error rate is thus given by the following integral for binary and quaternary DPSK

$$P_e = \frac{1}{2} - \frac{1}{2} \int_0^\infty \left(\prod_{\ell=1}^{L} \frac{1}{1 + xR_\ell} \right) e^{-(P+1)x} \, dx \tag{8-82}$$

and by the integral

$$P_e = \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2x}} \left(\prod_{\ell=1}^{L} \frac{1}{1 + xR_\ell} \right) e^{-(P+1)x} \, dx \tag{8-83}$$

for binary and quaternary PSK. The minor modifications for coherent and incoherent FSK are evident.

8.5 ERROR RATE ANALYSIS FOR MULTIPLE JAMMERS: CORRELATED JAMMER CHANNELS ONLY

In this section we generalize the previous analysis to the case in which the fading on the various jammer channels is correlated but the signal channel is still uncorrelated with the jammer channels. First, we wish to compute $H(Q)$, the d.f. of the random variable q [see Eqs.(8-64) - (8-66)]. This d.f. may be computed from (8-74) once the c.f. $C(s)$ is known. Under the assumption that the signal channel fluctuations are independent of the jammer channels' fluctuations, $C(s)$ factors into the product

$$C(s) = \overline{e^{-sr_0}} \cdot \overline{e^{s\sum_{\ell=1}^{L} r_\ell}} \tag{8-84}$$

The first average has already been computed [see Eq.(7.51)]. We consider the evaluation of the second average which is the c.f. of the random variable

$$d = \sum_{\ell=1}^{L} r_\ell = \sum_{\ell=1}^{L} |\mu_\ell|^2 \tag{8-85}$$

Comparing Eq.(8-20) with Eqs.(8-64) and (8-65) we can make the identification

$$\mu_\ell = z_\ell \sqrt{xn_\ell} \tag{8-86}$$

i.e.,

$$d = \sum_{\ell=1}^{L} xn_\ell \, |z_\ell|^2 \tag{8-87}$$

The random variable d is a quadratic form in correlated complex zero mean Gaussian variables. Turin [8.1] has evaluated the characteristic function of a quadratic form in complex Gaussian variables,

$$d = \underline{z}^H \underline{Q}\, \underline{z} \tag{8-88}$$

as

$$\phi(s) = \overline{e^{-sd}} = \frac{1}{\det[\, \underline{I} + s\underline{M}\, \underline{Q} \,]} \tag{8-89}$$

where $\underline{Q}$ is the matrix defining the quadratic form and $\underline{M}$ is the moment matrix of the random variables. In our case $\underline{Q}$ is the diagonal matrix

$$\underline{Q} = x\, \underline{N} \tag{8-90}$$

where

$$\underline{N} = \begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & n_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & & 0 & n_L \end{bmatrix} \tag{8-91}$$

and

$$\underline{M} = \{\, \overline{z_\ell^* z_n} \;\; ; \;\; \ell,n = 1,2,\ldots L \,\} \tag{8-92}$$

so that

$$\phi(s) = \frac{1}{\det[\, \underline{I} + s\underline{M}\, \underline{N} \,]} \tag{8-93}$$

Using (8-89) and (8-71) in (8-84)

$$C(s) = \frac{1/a_0}{s + 1/a_0} \frac{1}{\det[\underline{I} - sx\,\underline{M}\,\underline{N}]} \tag{8-94}$$

The d.f. H(Q) is thus given by (see (8-74)) the contour integral

$$H(Q) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{1/a_0}{s + 1/a_0} \frac{1}{\det[\underline{I} - sx\,\underline{M}\,\underline{N}]} \frac{1}{s} \, ds \tag{8-95}$$

It will be shown below that the poles of the second factor in (8-74) are all in the right half plane. Thus, using Cauchy's residue theorem,

$$H(Q) = 1 - \frac{1}{\det\left[\underline{I} + \frac{x}{a_0}\underline{M}\,\underline{N}\right]} e^{-Q/a_0} \tag{8-96}$$

Using the definition (8-32), (8-34), and (8-81),

$$F(x) = 1 - \frac{1}{\det[\underline{I} + x\,\underline{M}\,\underline{R}]} e^{-Px} \tag{8-97}$$

where

$$\underline{R} = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & & 0 & R_L \end{bmatrix} \tag{8-98}$$

A single integral expression for P_e is obtained by substituting (8-97) in (8-73) and using the appropriate expressions for $(-f(\rho))$. Thus

$$P_e = \frac{1}{2} - \frac{1}{2}\int_0^\infty \frac{e^{-(P+1)x}}{\det[\underline{I} + x\underline{M}\,\underline{R}]}\,dx \quad ; \quad \text{binary and quaternary DPSK} \tag{8-99}$$

$$P_e = \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2x}}\,\frac{e^{-(P+1)x}}{\det[\underline{I} + x\underline{M}\,\underline{R}]}\,dx \quad ; \quad \text{binary and quaternary DPSK} \tag{8-100}$$

Alternatively we note that the denominator of (8-93) is an L^{th} order polynomial in sx and may be factored to explicitly show the poles of $\phi(s)$. If the eigenvalues of the matrix product $\underline{M}\,\underline{N}$ are denoted by $\{\lambda_\ell; \ell=1,2,\ldots L\}$ then

$$\det[\underline{I} + x\underline{M}\,\underline{N}] = \prod_{\ell=1}^{L} (1 - sx\lambda_\ell) \tag{8-101}$$

Due to the positive definite nature of $\underline{M}\,\underline{N}$, all the eigenvalues are positive numbers and thus the poles of $\phi(s)$ are in the right half plane. Using (8-101) in (8-94) we arrive at the following expression for the c.f. of q

$$c(s) = \frac{1/a_0}{s + 1/a_0} \prod_{k=1}^{L} \left(\frac{-1/x\lambda_k}{s - 1/x\lambda_k} \right) \tag{8-102}$$

which is identical to the form (8-72) in the case of independently fluctuating jammer channels.

It follows that the d.f. $H(Q)$ is given by an analogous equation to (8-76)

$$H(Q) = 1 - \prod_{\ell=1}^{L} \frac{a_0}{a_0 + x\lambda_\ell} e^{-Q/a_0} \qquad (8\text{-}103)$$

Then using (8-77) and (8-79), the c.f. of ℓ becomes

$$F(x) = 1 - \left(\prod_{\ell=1}^{L} \frac{1}{1 + xS_\ell} \right) e^{-Px} \qquad (8\text{-}104)$$

where

$$S_\ell = \lambda_\ell / \rho_0 \qquad (8\text{-}105)$$

Finally, the error rate expressions are identical to (8-82) and (8-83) with the replacement $S_\ell = R_\ell$.

8.6 ERROR RATE ANALYSIS FOR MULTIPLE JAMMERS: GENERAL CORRELATED CHANNEL CASE

We consider here the general case of mutually correlated communicator and jammer channels. The d.f. $H(Q)$ can be computed from the c.f. $C(s)$ via (8-74). We may use the results of Turin [8.1] again to compute $C(s)$, since the latter is the c.f. of a quadratic form in complex Gaussian variables, i.e.,

$$C(s) = \overline{e^{-s\left(|\mu_0|^2 - \sum_{\ell=1}^{L} |\mu_k|^2\right)}} \tag{8-106}$$

From (8-88) and (8-89) we see that

$$C(s) = \frac{1}{\det\left[\underline{I} + s\underline{M}\,\underline{Q}\right]} \tag{8-107}$$

where now

$$\underline{M} = \{\overline{z_\ell^* z_n} \; ; \; \ell,n = 0,1,2,\ldots L\} \tag{8-108}$$

and Q is the diagonal matrix

$$Q = \begin{bmatrix} s_0 & 0 & 0 & \cdots & 0 \\ 0 & -xn_1 & 0 & \cdots & 0 \\ 0 & 0 & -xn_2 & & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 0 & -xn_L \end{bmatrix} \qquad (8\text{-}109)$$

From (8-74) we see that

$$H(Q) = \sum \mathrm{Res} \left\{ \frac{e^{sQ}}{s \det [\underline{I} + s\underline{M}\ \underline{Q}]} \right\}_{\text{left half plane}} \qquad (8\text{-}110)$$

where the pole at the origin is included in the left half plane by indentation. In the case of no correlation between the signal and jammer channels there was only one pole in the left half plane aside from the pole at the origin. It is not known at this point if this type of pole configuration will stay the same for arbitrary correlation. However, for small correlation at least, one may expect only one left half plane pole. If this pole is denoted by

$$s = -1/\sigma \qquad (8\text{-}111)$$

then, from (8-110),

$$H(Q) = 1 - \left\{ \frac{(1 + s\sigma)\ e^{-Q/\sigma}}{\det [\underline{I} + s\underline{M}\ \underline{Q}]} \right\}_{s = -1/\sigma} \qquad (8\text{-}112)$$

Alternatively if the eigenvalues of $\underline{M}\ \underline{Q}$ are $\{\lambda_\ell\ ;\ \ell=0,1,\ldots,L\}$ with $\lambda_0=\sigma$, then

$$H(Q) \quad = \quad 1 - \prod_{\ell=1}^{L} \frac{\sigma(x)}{\sigma(x) + \lambda_\ell(x)}\, e^{-Q/\sigma(x)} \qquad (8\text{-}113)$$

where we have shown explicitly the dependence of σ and λ_ℓ on x.

It follows from (8-79) that the d.f. of ρ can be taken as either of the two equations:

$$F(x) \quad = \quad 1 - \prod_{\ell=1}^{L} \frac{\sigma(x)}{\sigma(x) + \lambda_\ell(x)}\, e^{-px/\sigma(x)} \qquad (8\text{-}114)$$

$$F(x) \quad = \quad 1 - \left\{\frac{1 + s\sigma(x)}{\det\,[\underline{I} + s\underline{M}\ \underline{Q}]}\right\} \quad e^{-px/\sigma(x)} \qquad (8\text{-}115)$$

$$s = -1/\sigma(x)$$

A single integral expression for P_e is obtained by using either (8-114) or (8-115) in (8-73), with appropriate specification of $(-f(\rho))$ for the modem of interest.

We will now examine the case of a single jammer in more detail. In this case

$$M \quad = \quad \begin{bmatrix} 1 & r_{01} \\ r^*_{01} & 1 \end{bmatrix} \qquad (8\text{-}116)$$

$$Q \quad = \quad \begin{bmatrix} s_0 & 0 \\ 0 & -xn_1 \end{bmatrix} \qquad (8\text{-}117)$$

Simplifying the notation by the definitions

$$r_{01} = r \tag{8-118}$$

$$n_1 = n \tag{8-119}$$

we find that

$$P(s) = \det[\underline{I} + s\underline{M}\,\underline{Q}] = \det \begin{bmatrix} 1+ss_0 & -sxnr \\ sr^*s_0 & 1-sxn \end{bmatrix} \tag{8-120}$$

or

$$\begin{aligned} P(s) &= -xns_0(1 - |r|^2)\, s^2 + s(s_0 - xn) + 1 \\ &= (1 + s\sigma)(1 - s\lambda) \end{aligned} \tag{8-121}$$

where

$$\sigma = \frac{2xns_0(1 - |r|^2)}{\sqrt{(s_0 - xn)^2 + 4xns_0(1-|r|^2)} - (\rho_0 - xn)} \geq 0 \tag{8-122}$$

$$\lambda = \frac{2xns_0(1 - |r|^2)}{\sqrt{(s_0 - xn)^2 + 4xns_0(1-|r|^2)} + (s_0 - xn)} \geq 0 \tag{8-123}$$

Using (8-122) and (8-123) in (8-114),

$$F(x) = 1 - \frac{1}{2}\left(1 + \frac{s_0 - xn}{\sqrt{(s_0 + xn)^2 - 4xns_0|r|^2}}\right) e^{-px/\sigma(x)} \tag{8-124}$$

With the definitions P and R in (8-34) and (8-35),

$$F(x) = 1 - \frac{1}{2}\left(1 + \frac{1 - xR}{\sqrt{(1 + xR)^2 - 4Rx|r|^2}}\right)\exp\left(-P\,\frac{\sqrt{(1+xR)^2 - rRx|r|^2} - (1-xR)}{2R(1-|r|^2)}\right) \tag{8-125}$$

8.7 SOME EXAMPLES

For purposes of illustration we present some numerical results for the simplest case - a single jammer and a single auxiliary antenna. We examine the case of PSK transmission. From Eq.(4-24) the error rate is given by

$$P_e = \frac{1}{2} - \sqrt{\frac{\pi}{R}} \; e^{\left(\frac{1+P}{R}\right)} \; Q\sqrt{2\left(\frac{1+P}{R}\right)} \tag{8-126}$$

To evaluate P_e it is necessary to determine P and R. For the case of one antenna and one jammer, these are given by (see Eqs. (4-4),(4-5),(3-4) and (3-6)

$$P = \frac{1 + |p_1|^2}{s_0} \tag{8-127}$$

$$R = \frac{J_1}{N} \; \frac{\overline{|h_1|^2}}{s_0} \tag{8-128}$$

where p_1 is the optimum complex weight, s_0 is the value of average energy/bit /noise power density for the received signal, N is the thermal noise power of the main and auxiliary antenna outputs and J_1 is the transmittted jamming power. $h_1(t)$ is the complex gain of the residual channel from the jammer to the canceller output, (Eq. (2-3))

$$h_1(t) = h_{10}(t) - p_1 h_{11}(t) \tag{8-129}$$

in which $h_{10}(t)$, $h_{11}(t)$ are the jammer to main and auxiliary antenna channel complex gains (Eq.(2-6)) respectively.

The optimum weight is obtained from equations in Section 3. For the single weight case we find that

$$p_1 = \frac{J_1 \overline{h_{10}h_{11}^*}}{\overline{|h_{11}|^2} J_1 + N} \tag{8-130}$$

Using (8-130) in (8-129) and computing $\overline{|h_1|^2}$ we find that

$$\overline{|h_1|^2} = \overline{|h_{10}|^2} - \frac{\overline{|h_{10}h_{11}^*|^2} J_1}{\overline{|h_{11}|^2} J_1 + N}\left(1 + \frac{N}{\overline{|h_{11}|^2} J_1 + N}\right) \tag{8-131}$$

Using the definitions

$$G_{10} = |A_0(\theta_1,\phi_1)|^2 \tag{8-132}$$

$$G_{11} = |A_1(\theta_1,\phi_1)|^2 \tag{8-133}$$

$$\rho = \overline{g_{10}g_{11}^*} \tag{8-134}$$

for the antenna gains in the direction of the jammer and using Eq.(8-131) and (8-132)

$$\overline{|h_1|^2} = G_{10} - \frac{G_{10}G_{11} |\rho|^2 J_1}{G_{11}J_1 + N}\left(1 + \frac{N}{G_{11}J_1 + N}\right) \tag{8-135}$$

Three parameters of interest are †

$$\beta = \frac{G_{11}J_1}{N} \tag{8-136}$$

$$\gamma = \frac{G_{10}}{G_{11}} \tag{8-137}$$

$$\delta = 1 - |\rho|^2 \tag{8-138}$$

where β is the ratio of the received jammer power to thermal noise power in the auxiliary antenna output, γ is the ratio of the main beam antenna gain to the auxiliary antenna gain in the direction of the jammer, and δ is a decorrelation parameter.

Using these parameter definitions in (8-135) and (8-128) we find that

$$R = \frac{\gamma}{s_0} \frac{(\beta)(\beta+2)}{(\beta+1)^2} (1 + \delta\rho) \tag{8-139}$$

Also using these parameter definitions and (8-132) through (8-133) in (8-130) and (8-127) we see that

$$P = \frac{1}{s_0} \left[1 + \left(\frac{\beta}{\beta+1}\right)^2 \gamma(1-\delta) \right] \tag{8-140}$$

There are now four parameters that define the performance of the system: s_0, δ, γ, and β.

---

† The decorrelation parameter δ is related to ε in Eq. (6-42) by $\delta = 2\varepsilon+\varepsilon^2 \approx 2\varepsilon$ and to y in Eq. (6-45) by $\delta \approx 2y^2$.

Some limiting cases are of interest. Consider first the case of no decorrelation, $\delta = 0$, then

$$R = \frac{\gamma}{s_0} \frac{(\beta)(\beta+2)}{(\beta+1)^2} \quad ; \quad \gamma = 0 \tag{8-141}$$

$$P = \frac{1}{s_0} \left[1 + \gamma \left(\frac{\beta}{\beta+1} \right)^2 \right] \quad ; \quad \gamma = 0 \tag{8-142}$$

Assume also a large jammer, i.e., $\beta >> 1$, then

$$R \rightarrow \frac{\gamma}{s_0} \tag{8-143}$$

$$P \rightarrow \frac{1 + \gamma}{s_0} \tag{8-144}$$

$$\frac{1 + D}{R} \rightarrow \frac{s_0 + 1 + \gamma}{\gamma} \tag{8-145}$$

Typical values of γ are of the order of 1 or less so that at large SNR's $(1+P)/R$ is a large number. Thus from Eq.(4-27)

$$P_e \rightarrow \frac{1 + 2\gamma}{4s_0} \quad ; \beta >> 1, \; s_0 >> 1 \tag{8-146}$$

The advisability of keeping γ small is evident from (8-146), the smaller γ yields the smaller error rate. It is also evident that with G=0 and γ small, the jammer is effectively cancelled and the same performance is achieved as with no jammer.

Consider now the limiting case δ=1, complete decorrelation. Then,

$$R \rightarrow \frac{q}{s_0}\beta \qquad ; \qquad \beta >> 1,\ \delta = 1 \tag{8-147}$$

$$P = \frac{1}{s_0} \qquad ; \qquad \delta = 1$$

To the extent that $\gamma\beta/s_0$ is still a small number and s_0 is larger, we may still use the approximation in Eq.(4-6),

$$P_e \rightarrow \frac{1 + \gamma\beta}{4s_0} \tag{8-148}$$

Now we see that the jammer can increase the error rate considerably. Under the condition δ = 1, the auxiliary antenna gain goes to zero (p_1=0) and no sidelobe cancellation is possible.

Figure 8.2 presents a representative calculation of error rate as a function of δ for β=1000, γ=1 with a family parameter s_0=3n dB, n=1,2,...9. Note the rapid degradation in error rates with δ at the high SNR's. Typically 3 dB of degradation sets in when δ increases from 0 to .003 .

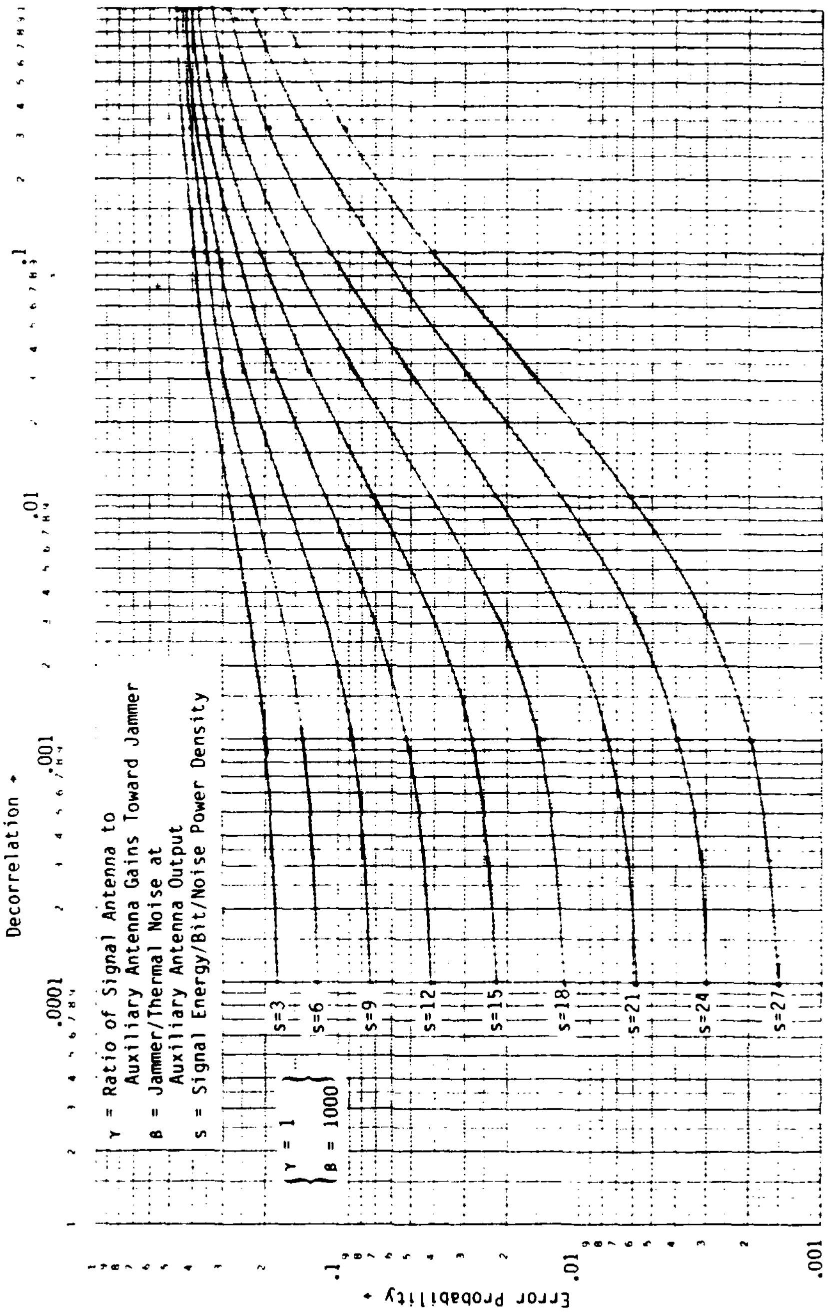

Figure 8.2 Error Probability for PSK as a Function of Decorrelation Parameter δ for Values of s = 3n dB, n=1,2,...9 and β = 1000, γ=1. Single jammer and auxiliary antenna.

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APPENDIX

PROOF OF A MATRIX IDENTITY

We shall prove a matrix identity that was useful in the text. Given a symmetric matrix

$$r_M = \{ \rho_{mp} \; ; \; m,p = 1,2,\ldots,M \} \tag{A-1}$$

of the form

$$\rho_{mp} = \rho^*_{pm} = R_{m-p} \tag{A-2}$$

i.e., Hermitian symmetric with the property that a matrix element depends for its value only on the difference between its indices, then the following matrix property is valid,

$$\underline{\rho}_M^H \underline{r}_M^{-1} \rho_M = 1 - \frac{\det \underline{r}_{M+1}}{\det \underline{r}_M} \tag{A-3}$$

To prove (A-3) we note first that $\underline{r}_{M+1}$ can be partitioned in the form

$$\underline{r}_{M+1} = \left[\begin{array}{c|c} 1 & \underline{\rho}_M^T \\ \hline \underline{\rho}_M^* & \underline{r}_M \end{array} \right] \tag{A-4}$$

where we have normalized

$$R_0 = 1 \tag{A-5}$$

The determinant of $\underline{r}_{M+1}$ is equal to the sum of the products of the elements of any single row, such as the first row, by their cofactors. If the cofactors of the first row of $\underline{r}_{M+1}$ are denoted by $[A_{11}, A_{12}, \ldots A_{1,M+1}]$, then we see that

$$A_{11} = \det r_N \tag{A-6}$$

$$A_{1m} = (-1)^{1+m} \det \left| \begin{bmatrix} \underline{\rho}_M^* & \underline{C}_1 \cdots & \underline{C}_{m-2} & C_m & \cdots & \underline{C}_M \end{bmatrix} \right| \quad ; \quad 3 < m < M-1 \tag{A-7}$$

$$A_{12} = \det \left| \begin{bmatrix} \underline{\rho}_M^* & \underline{C}_2 & \cdots & \underline{C}_M \end{bmatrix} \right| \tag{A-8}$$

$$A_{1,M+1} = (-1)^{M+2} \det \left| \begin{bmatrix} \underline{\rho}_M^* & \underline{C}_1 & \underline{C}_2 & \cdots & \underline{C}_{M-1} \end{bmatrix} \right| \tag{A-9}$$

where $\underline{C}_p$ *is the* p^{th} *column vector in* $\underline{r}_M$, *i.e.*,

$$\underline{r}_M = \left| \begin{bmatrix} \underline{C}_1 & \underline{C}_2 & \cdots & \underline{C}_M \end{bmatrix} \right| \tag{A-10}$$

Note that the matrix in (A-7) differs from that in (A-8) in that the m^{th} column of r_M has been removed and a new first column ρ_M^* inserted. If two columns of a matrix are interchanged, the determinant of the new matrix is the negative of the determinant of the original matrix. Thus A_{1m} can be expressed in the alternate form

$$A_{1m} = -\det\left[\underline{C}_1, \underline{C}_2 \cdots \underline{C}_{m-2}\ \underline{\rho}^*_M\ C_m \cdots \underline{C}_M\right] \quad ; \quad 3 \le m \le M \tag{A-11}$$

$$A_{1,m+1} = -\det\left[\underline{C}_1, \underline{C}_2 \cdots \underline{C}_{M-1}, \underline{\rho}^*_M\right] \tag{A-12}$$

where we have interchanged the first column in (A-7) with the second column, and then the second column with the third, etc., until $\underline{\rho}^*_M$ interchanges position with $\underline{C}_{m-2}$, i.e., m-2 interchanges for a factor of $(-1)^{m-2}$.

If we define the vector

$$\underline{B} = \begin{bmatrix} -A_{12} \\ -A_{13} \\ \vdots \\ -A_{1,M+1} \end{bmatrix} \tag{A-13}$$

and evaluate the determinant of $\underline{r}_{M+1}$ by use of the cofactors of the first row, it follows that

$$\det r_{M+1} = \det r_M - \underline{B}^T \underline{\rho}_M \tag{A-14}$$

Examination of the structure of $\{ A_{1m} ; m=2,\ldots,M+1 \}$ reminds one immediately of a rule for the solution of a set of linear equations. In particular, consider the equation for the unknown vector $\underline{x}$:

$$\underline{r}_M \underline{x} = \underline{\rho}^*_M \tag{A-15}$$

The solution for $\underline{x}$ is given by

$$\underline{x} = \underline{r}_M^{-1} \underline{\rho}^*{}_M \tag{A-16}$$

We denote x_p as the p^{th} component of $\underline{x}$,

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \tag{A-17}$$

The solution for x_p is given by

$$x_p = \frac{\det \left[\underline{C}_1, \underline{C}_2, \ldots \underline{C}_{p-1}, \underline{\rho}^*_M \, \underline{C}_{p+1} \ldots \underline{C}_M \right]}{\det r_M} \tag{A-18}$$

Comparison of (A-11) and (A-13) with (A-17) and (A-18) shows that

$$\underline{B}^T = x^T \det \underline{r}_M = \underline{\rho}_M^H \, \underline{r}_M^{-1} \det \underline{r}_M \tag{A-19}$$

where we have used the fact that r_M is Hermitian symmetric.

Then, using (A-19) and (A-14) and dividing the equation by det r_M we obtain Eq.(A-3).

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